D-MATH
HS 2019
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Exercise Sheet 8

Commutative Algebra

Let $A \neq 0$ be a commutative ring.

1 Let $P \in A[T]$ be a monic polynomial of degree n.

a. Consider the quotient ring $B = A[T]/(P)$. Prove that B is integral over A and that there exists $b \in B$ such that $P(b) = 0$.

b. Prove that B is a free A-module of rank n. In particular the natural morphism $A \to B$ is injective.

c. In the ring $C = A[X_1, \dots, X_n]$ let I be the ideal generated by the coefficients of the polynomial

$$P - \prod_{i=1}^{n}(T - X_i) \in C[T].$$

Prove that the ring C/I is a free A-module of rank $n!$. This ring is called the *splitting algebra* of P.

d. Assume that A is a field and let $\mathfrak{m}$ be a maximal ideal of C. Prove that the field $C/\mathfrak{m}$ is a splitting field of P.

2 Let A be an integral domain, K its field of fractions.

a. Let $x \in K$ be integral over A. Show that there exists $a \in A$, $a \neq 0$ such that $ax^n \in A$ for every $n \in \mathbb{N}$.

b. Assume that A is a noetherian ring. Let $a \in A$, $a \neq 0$ and $x \in K$ such that $ax^n \in A$ for every $n \in \mathbb{N}$. Prove that x is integral over A.

3

a. Let A be an integral domain and let $t \in A$ be such that A/tA has no nilpotent elements except for 0. Assume also that the ring of fractions A_t is integrally closed. Prove that A is integrally closed.

b. Prove that the ring $\mathbb{C}[X,Y,Z]/(XZ - Y(Y+1))$ is integrally closed (use a. with $t = X$).

4 Let $A \subseteq B$ be a ring extension, with A integrally closed in B.

a. Let $P, Q \in B[T]$ be monic polynomials such that $PQ \in A[T]$. Show that $P, Q \in A[T]$.
Hint: Use exercise 1 to introduce a ring $C \supseteq B$ such that $P = \prod_{i=1}^{m}(T - a_i)$ and $Q = \prod_{i=1}^{n}(T - b_i)$ where $a_i, b_j \in C$.

b. Show that $A[T]$ is integrally closed in $B[T]$.
Hint: If $P \in B[T]$ is integral over $A[T]$, consider the monic polynomial $Q = T^m + P$, for m large enough.

5 Let K be a field and let $\underline{v} = (v_1, \dots, v_r) \in \mathbb{Z}^r$ and consider the monomial

$$Y^{\underline{v}} := Y_1^{v_1} \dots Y_r^{v_r}.$$

Let B be the K-algebra

$$B = K[Y^{\underline{v_1}}, \dots, Y^{\underline{v_m}} : \underline{v_i} \in \mathbb{Z}^r].$$

Let M be the $r \times m$ matrix with coefficients in $\mathbb{Q}$ with $\underline{v_i}$ as columns. Show that $\dim A = \operatorname{rank} M$.

6 Assume that A is noetherian. Show that

$$\dim A = \max\{\dim(A/\wp) : \wp \text{ minimal prime of } A\}.$$