D-MATH
HS 2019
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Exercise Sheet 9

Commutative Algebra

Let $A \neq 0$ be a commutative ring.

1 Let M, N be A-modules and let $f : M \to N$ be a morphism. Assume that M has finite length.

a. Show that $\ker f$ and $\operatorname{im} f$ have finite lengths and that

$$\ell(\operatorname{im} f) + \ell(\ker f) = \ell(M).$$

b. Assume that $N = M$. Show that the following conditions are equivalent: (i) f is bijective; (ii) f is injective; (iii) f is surjective.

c. Assume that M is artinian. Show that there exists an integer $n \geq 1$ such that $\ker(f^n) + \operatorname{im}(f^n) = M$.

2 Let M be an A-module of finite length and let u be an endomorphism of M.

a. Show that there exists a smallest integer $p \geq 0$ such that

$$\ker(u^n) = \ker(u^p)$$

for any $n \geq p$.

b. Show that there exists a smallest integer $q \geq 0$ such that

$$\operatorname{im}(u^n) = \operatorname{im}(u^q)$$

for any $n \geq q$.

c. Show that $p = q$.

d. Show that $\ker(u^p)$ and $\operatorname{im}(u^p)$ are direct summands in M.

3 Consider the Nakayama's Lemma as given by Corollary 2.5 of Atiyah-MacDonald's book. Show that one cannot omit the hypothesis that the module be finitely generated (consider the $\mathbb{Z}$-module $\mathbb{Q}$).

4 Let I be a finitely generated ideal of A such that $I = I^2$. Show that there exists $e \in A$ such that $e = e^2$ and $I = (e)$.
Hint: Apply Nakayama's Lemma to find an element $a \in I$ such that $(1 + a)I = 0$.

5 a. Let M be an A-module of finite length. Show that the canonical morphism

$$M \longrightarrow \prod_{\mathfrak{m} \subseteq A \text{ maximal}} M_{\mathfrak{m}},$$

sending $x \in M$ to the family of fractions $x/1$, is an isomorphism of A-modules.

b. Assume that A is artinian. Show that the canonical morphism

$$A \longrightarrow \prod_{\mathfrak{m} \subseteq A \text{ maximal}} A_{\mathfrak{m}}$$

is an isomorphism of rings. Hence any commutative artinian ring is a product of local rings.