D-MATH
HS 2019
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Solutions 10

Commutative Algebra

1 Let $\mathfrak{p} \supset \mathfrak{p}_1 \supset \mathfrak{p}_0$ with $\mathfrak{p}$ not contained in any ideal of S. By the prime avoidence lemma, pick an element

$$a \in \mathfrak{p} - (\mathfrak{p}_0 \cup \bigcup_{\mathfrak{p}' \in S} \mathfrak{p}').$$

Pick also a minimal prime $\mathfrak{p}_1'$,

$$(a) + \mathfrak{p}_0 \subseteq \mathfrak{p}_1' \subseteq \mathfrak{p}.$$

Then $\mathfrak{p}_1'/\mathfrak{p}_0$ is a minimal prime ideal of $((a)+\mathfrak{p}_0)/\mathfrak{p}_0$ in $A/\mathfrak{p}_0$. By the Hauptidealsatz, since $((a)+\mathfrak{p}_0)/\mathfrak{p}_0$ is principal, $\mathrm{ht}(\mathfrak{p}_1'/\mathfrak{p}_0) = 1$. But the chain

$$\mathfrak{p}/\mathfrak{p}_0 \supset \mathfrak{p}_1/\mathfrak{p}_0 \supset \mathfrak{p}_0/\mathfrak{p}_0$$

shows that $\mathfrak{p}/\mathfrak{p}_0$ has height at least 2. Therefore $\mathfrak{p} \supset \mathfrak{p}_1' \supset \mathfrak{p}_0$ are distinct primes, and $\mathfrak{p}_1' \notin S$ because it contains a.

For the general case, apply the special case to $\mathfrak{p} \supset \mathfrak{p}_{d-1} \supset \mathfrak{p}_{d-2}$ to obtain a chain $\mathfrak{p} \supset \mathfrak{p}_{d-1}' \supset \mathfrak{p}_{d-2}$ such that $\mathfrak{p}_{d-1}'$ is not contained in any ideal of S. Apply again the special case to $\mathfrak{p}_{d-1}' \supset \mathfrak{p}_{d-2} \supset \mathfrak{p}_{d-1}$ and repeat the argument.

2 The first inequality follows immediately from the correspondence between ideals in A and in $A/\mathfrak{a}$.

Let $d := \mathrm{ht}(\mathfrak{p}/\mathfrak{a})$. Then there exist elements $a_1, \ldots, a_d$ in A such that $\mathfrak{p}/\mathfrak{a}$ is a minimal prime ideal of $(\overline{a}_1, \ldots, \overline{a}_d) \subseteq A/\mathfrak{a}$ (as a consequence of the Hauptidealsatz). Let $\mathfrak{a} = (b_1, \ldots, b_n)$. Then $\mathfrak{p}$ is a minimal prime ideal of $(a_1, \ldots, a_d, b_1, \ldots, b_n)$, hence $\mathrm{ht}(\mathfrak{p}) \le d + n$.

3

a. If $\dim A = 0$, then it is regular if and only if its maximal ideal can be generated by the empty set, and so is zero. This means that A is a field.

b. Consider $\mathfrak{m}' := \mathfrak{m}/(a)$, the maximal ideal of $A/(a)$. Then by exercise 2

$$\mathrm{ht}(\mathfrak{m}') \le \mathrm{ht}(\mathfrak{m}) \le \mathrm{ht}(\mathfrak{m}') + 1.$$

Therefore

$$\begin{aligned}\dim_k(\mathfrak{m}'/\mathfrak{m}'^2) &\geq \mathrm{ht}(\mathfrak{m}')\\ &\geq \mathrm{ht}(\mathfrak{m})-1\\ &= \dim_k(\mathfrak{m}/\mathfrak{m}^2)-1\\ &= \dim_k(\mathfrak{m}'/\mathfrak{m}'^2).\end{aligned}$$

Equality must hold throughout.

c. Note that for every $a \in \mathbb{Q}^\times$,

$$(X+a)(1/a^2X - 1/a) \equiv 1$$

in A. In particular $(X+a) \in A^\times$. This proves that A is local with maximal ideal (X). Also, (X) is a minimal prime of (X^2), which proves $\dim A = 0$; but $\mu((X)) = 1 > 0$.

d. A complete proof of this fact can be found in Atiyah-MacDonald, page 123.

e. Let $(x, y) := Q$ and consider the $\mathbb{Q}$-vector spaces morphism

$$\begin{aligned}\theta_Q : \mathbb{Q}[X,Y] &\longrightarrow \mathbb{Q}^2\\ g &\longmapsto (\partial_X g_{|Q}, \partial_Y g_{|Q}).\end{aligned}$$

It's easy to see that $\theta_Q(P)$ generates $\mathbb{Q}^2$ and $\theta_Q(P^2) = 0$. Then we can consider the induced map (always denoted by θ_Q)

$$\theta_Q : P/P^2 \longrightarrow \mathbb{Q}^2.$$

Since $\dim_{\mathbb{Q}}(P/P^2) = 2$, θ_Q is an isomorphism of $\mathbb{Q}$-vector spaces. Note also that

$$\dim_{\mathbb{Q}}((f)+P^2/P^2) = \dim_{\mathbb{Q}}(\theta_Q((f))) = \begin{cases}1 & \text{if } (\partial_X f_{|Q}, \partial_Y f_{|Q}) \neq (0,0)\\ 0 & \text{otherwise}\end{cases}.$$

In the following we'll denote by P both the ideal of $\mathbb{Q}[X,Y]$ and its projection in B. One has

$$\begin{aligned}PB_P/(PB_P)^2 &\simeq (P/P^2)\otimes_B B_P\\ &\simeq P\otimes_B (B/P)_P\\ &\simeq P/P^2\\ &\simeq P/((f)+P^2).\end{aligned}$$

By the $\mathbb{Q}$-isomorphism

$$P/P^2 \simeq ((f)+P^2)/P^2 \oplus P/((f)+P^2)$$

we see that

$$\dim_{\mathbb{Q}}(\theta_Q((f))) = 2 - \mu(PB_P).$$

On the other hand

$$1 = \operatorname{ht}(f) = 2 - \dim B = 2 - \dim B_P,$$

so

$$\dim_{\mathbb{Q}}(\theta_Q((f))) = \operatorname{rank}((\partial_X f_{|Q}, \partial_Y f_{|Q})) = 1 \iff B_P \text{ is regular.}$$

4 The elements $1/p^n$ generate $\mathbb{Z}[1/p]$ over $\mathbb{Z}$. Every proper subgroup of $\mathbb{Z}[1/p]/\mathbb{Z}$ is of the form $N/\mathbb{Z}$ for a proper subgroup $N \subset \mathbb{Z}[1/p]$ containing $\mathbb{Z}$. Thus for some $n \geq 0$, $1/p^{n+1}$ does not lie in N. A fortiori, for all $m > n$, $1/p^m \notin N$. Consider now an arbitrary $a/p^m \in N$, with $p \nmid a$. Pick x and y so that

$$1 = xp^m + ya.$$

Thus

$$1/p^m = x + y(a/p^m) \in N,$$

and so $m \leq n$. In particular $N/\mathbb{Z} \subset \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ is finite. We proved that every subgroup of $\mathbb{Z}[1/p]/\mathbb{Z}$ is finite.
Any descending chain of subgroups of $\mathbb{Z}[1/p]/\mathbb{Z}$, after the first non-trivial step there can be only finitely many other non-trivial steps for cardinality reasons. Thus $\mathbb{Z}[1/p]/\mathbb{Z}$ is artinian.
It is not noetherian because the submodules $N_n := p^{-n}\mathbb{Z}/\mathbb{Z}$ form an ascending chain of submodules $0 = N_0 \subset N_1 \subset \ldots$ which doesn not become stationary.