D-MATH
HS 2019
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Solutions 11

Commutative Algebra

1 Denote by x, y and z the cosets of X, Y and Z in B. Then B is generated as $\mathbb{C}$-algebra by x, y, z. They satisfy $f(x, y, z) = x^2y - xy^3 + z^4 - 1 = 0$ and $xy + z^2 = 0$. Let $y_1 := x - z$, $y_2 := y - z$, $y_3 := z$. Then

$$f(y_1 + y_3, y_2 + y_3, y_3) = 0$$

and since f is monic in $y_3 = z$, y_3 is integral over $\mathbb{C}[y_1, y_2]$. Hence consider the integral extension $\mathbb{C}[y_1, y_2] \hookrightarrow \mathbb{C}[y_1, y_2, y_3] = B$.

2 a. You could also prove it directly, but it follows from the Weak Nullstellensatz: because f is not constant, it is not invertible and therefore is contained in a maximal ideal, which is the ideal of polynomials which are zero evaluated at p for some $p \in V(f)$.

b. Assume f is reducible. Since $\deg f = 2$, $f = g_1g_2$ for linear polynomials g_1 and g_2. The rows of the matrix M are the coefficients of X, Y and the constant coefficient in $\partial f/\partial X$, $\partial f/\partial Y$ and $2f - X\partial f/\partial X - Y\partial f/\partial X$. Expanding this in g_1 and g_2, all three are constant linear combinations of g_1 and g_2; thus the three rows are linearly independent.

c. Assume $f = g^2$. By the same argument as above, the three rows of M are the coefficients of constant multiples of g so that M has rank at most 1.

3 Let's proceed by induction on n. The base $n = 1$ is obvious. Let now $n > 1$, and note that

$$xA/x^nA \simeq A/x^{n-1}A;$$

hence one has an exact sequence

$$0 \longrightarrow A/x^{n-1}A \longrightarrow A/x^nA \longrightarrow A/xA \longrightarrow 0.$$

By properties of associated primes, and by induction, we get

$$\mathrm{Ass}(A/x^nA) \subseteq \mathrm{Ass}(A/xA) \cup \mathrm{Ass}(A/x^{n-1}A) = \mathrm{Ass}(A/xA).$$

For the other inclusion, if we pick an associated prime $\wp$ of A/xA, it is equivalent to consider an inclusion

$$A/\wp \hookrightarrow A/xA.$$

Since x is not a zero-divisor, the map $A/xA \to A/x^nA$ is injective, so by composing we get

$$A/\wp \hookrightarrow A/x^nA,$$

which means that $\wp$ is an associated prime to A/x^nA.