D-MATH
HS 2019
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Solutions 12

Commutative Algebra

(1) See "(Mostly) Commutative Algebra" by Antoine Chambert-Loir, Proposition 5.4.13.

(2) See "(Mostly) Commutative Algebra" by Antoine Chambert-Loir for a more general situation, Theorem 5.5.9, part b.

(3) a. The minimal primary decomposition of I is

$$I = (X, Y) \cap (Z, W).$$

Hence there are two associated primes: (X, Y), (Z, W).

b. The minimal primary decomposition of I is

$$I = (X) \cap (X^2, Y).$$

Hence there are two associated primes: (X), (X, Y).

c. This case is a little bit more complicated than the previous ones, since I is not generated by monomials. Note that the primes (X, Y) and (Z, W) are minimal primes. A primary decomposition is given by

$$I = (X, Y) \cap (Z, W) \cap (J + (X, Y, Z, W)^3)$$

(the third primary component is embedded. It is not unique. We can replace the third power of the maximal ideal by any higher power and get the same intersection). In general for two ideals $\mathfrak{a}$ and $\mathfrak{b}$, one has $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$. The radical of the ideal I is the intersection of the two minimal primes:

$$\sqrt{I} = (XW, XZ, YW, YZ).$$

Therefore

$$\begin{aligned} \sqrt{I + (X, Y, Z, W)^3} &= \sqrt{(XW, XZ, YW, YZ) + (X, Y, Z, W)} \\ &= (X, Y, Z, W). \end{aligned}$$

The associated primes are then (X, Y), (Z, W) and (X, Y, Z, W).

4

a. Let J be $\wp$-primary such that $J \supseteq \wp^k$. The claim is that $J \supseteq \wp^{(k)}$. Let $a \in \wp^{(k)}$, then by definition there exists $s \in A - \wp$ so that $as \in \wp^k \subseteq J$. This implies that $a \in J$ or $s \in \sqrt{J} = \wp$, which is clearly not possible.

b. Let $\sum a_i b_i \in \wp^{(k)}\wp^{(h)}$, with i, j in a finite set of indices, and $a_i \in \wp^{(k)}$, $b_i \in \wp^{(h)}$. Pick $s_i, s_i' \notin \wp$ such that $s_i a_i \in \wp^{(k)}$, $s_i' b_i \wp^{(h)}$. Then $s_i s_i' a_i b_i \in \wp^k \wp^h = \wp^{k+h}$.

c. If $a \in A$ is so that sa is a multiple of p^k for some $s \notin pA$, then by unique factorization a must be a multiple of p^k, so $\wp^{(k)} \subseteq \wp^k$, then $\wp^{(k)} = \wp^k$.

d. Let $f \in A = \mathbb{C}[X_1, \ldots, X_n]$ and $g \notin \wp$ such that $fg \in \wp^k$. A polynomial h of $\wp^k$ is characterized by

$$\frac{\partial^{k-1} h}{\partial X_j^{k-1}}\bigg|_{(0,\ldots,0)} = 0 \quad \forall j = 1, \ldots, n.$$

Write $g = g_0 + g_1$, with $g_0 \in \mathbb{C}$, $g_0 \neq 0$ and $g_1 \in \wp$. Therefore, for each $j = 1, \ldots, n$

$$\begin{aligned} & g_0 \frac{\partial^{k-1} f}{\partial X_j^{k-1}} + \frac{\partial^{k-1}(f g_1)}{\partial X_j^{k-1}} = 0 \\ \Longrightarrow \quad & \frac{\partial^{k-1} f}{\partial X_j^{k-1}}\bigg|_{(0,\ldots,0)} = -\frac{1}{g_0} \frac{\partial^{k-1}(f g_1)}{\partial X_j^{k-1}}\bigg|_{(0,\ldots,0)} = 0 \end{aligned}$$

since $fg_1 \in \wp$. Then $f \in \wp^k$.

e. This follows by part a.