D-MATH
HS 2019
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Solutions 13

Commutative Algebra

(1) b. Let V be a discrete valuation ring of $k(X)$ for some discrete valuation v, with $k \subseteq V \subset k(X)$. Let $\mathfrak{m}$ be the maximal ideal of V. If $X \in V$, then $k[X] \subseteq V$. Set $\wp := \mathfrak{m} \cap k[X]$. Then $\wp$ is a prime ideal of $k[X]$. Moreover $k[X]_\wp \subseteq V$ since V is local. If $\wp = 0$ one has $k[X]_\wp = k(X)$, but we are assuming the strict inclusion. Hence $\wp \neq 0$, so $\wp = (f)$ for some $f \in k[X]$ irreducible. Because $k[X]_{(f)}$ is a DVR, by the below lemma $V = k[X]_{(f)}$, so the valuation v is equivalent to v_f, since they have the same valuation ring.
Suppose now $X \notin V$. Then $X^{-1} \in V$, so $k[X^{-1}] \subseteq V$. Moreover $X \notin V$ means X^{-1} is not a unit in V. So $\mathfrak{m} \cap k[X^{-1}] \supseteq (X^{-1})$, then $\mathfrak{m} \cap k[X^{-1}] = (X^{-1})$ since (X^{-1}) is maximal in $k[X^{-1}]$. Therefore $k[X^{-1}]_{(X^{-1})} \subseteq V$, and again by the lemma below $V = k[X^{-1}]_{(X^{-1})}$, so v is equivalent to v_∞.

Lemma. *Let $A \subseteq B$ be discrete valuation rings, with $L = frac(A) = frac(B)$. If $A \subseteq B \subset L$, then $A = B$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A, and let v be the valuation of L with valuation ring A. Let $\pi \in A$ be a uniformizer, that is $v(\pi) = 1$. Take $b \in B$ and assume $v(b) = n$. If $n \geq 0$, then $b \in A$. If $n < 0$, then $v(b\pi^{-n}) = 0$. So $u := b\pi^{-n} \in A^\times$.

Thus $\pi^{-1} = u^{-1}b\pi^{-1-n} \in B$ since $u^{-1} \in A \subseteq B$, $b \in B$ and $-1-n \geq n$, so $\pi^{-1-n} \in A$. But since each nonzero element of L can be written as a unit in A times a power of π, $B = L$, a contradiction to the assumption. □

(2) a. Let S be a multiplicative system. The localization $S^{-1}A$ is normal, because A is. Also, any nonzero localization of a noetherian domain is a noetherian domain. It remains to show that $S^{-1}A$ is either a field or of dimension 1. Suppose it is not a field. Then there exists $0 \neq f \in A - S$. Thus $(f/1) \subseteq S^{-1}A$ is a nonzero proper ideal, thus contained in a maximal ideal. Thuis shows that $\dim(S^{-1}A) \geq 1$. The other inequality holds in general by the one-to-one correspondence of prime ideals of $S^{-1}A$ and prime ideals of A which have empty intersection with S.

b. A is isomorphic to $\mathbb{C}[X, \frac{1-X^4}{X}] = \mathbb{C}[X, X^{-1}] = \mathbb{C}[X]_{(X)}$, which is Dedekind by part a.

c. A is not Dedekind because it has Krull dimension 2.

d. A is not Dedekind because it is not normal, for example, the element $Y/X \in \mathrm{frac}(A) - A$ is integral over A: $(Y/X)^3 - X = 0$.

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a. This is clear, since we require that for a nonzero element of $f \in K$ there is a minimum exponent of t occurring in f.

b. It is obviuos that K is a ring. A power series with nonzero constant coefficient is invertible (you can easly check it using the definition), then a multiplicative inverse of $t^n g$ as in part a. is $t^{-n} g^{-1}$.