D-MATH
HS 2019
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# Solutions 2

Commutative Algebra

(1) Assume $(A, I)$ is a local ring. Let $a \in A - A^\times$. Then $(a)$ is a proper ideal of $A$, hence it is contained in $I$ which is the only maximal ideal of $A$.

Conversely, let $\mathfrak{m}$ be another maximal ideal of $A$. Clearly $\mathfrak{m} \subseteq A - A^\times$, so $\mathfrak{m} \subseteq I$. But $I$ is maximal, then $\mathfrak{m} = I$.

(2) See Proposition 1.7.2 of Antoine Chambert-Loir, "(Mostly) Commutative Algebra".

(3) See Proposition 2.2.12 of Antoine Chambert-Loir, "(Mostly) Commutative Algebra".

(4) Denote by $\mathfrak{N} = \sqrt{(0)} = \bigcap\limits_{\wp \subseteq A} \wp$ the nilpotent radical of $A$.

(a.$\Rightarrow$b.) If $\wp$ is the only prime ideal of $A$, then clearly it is the only maximal ideal of $A$. Every non-unit of $A$ is then in $\wp = \mathfrak{N}$.
(b.$\Rightarrow$c.) The non-zero elements of $A/\mathfrak{N}$ are the elements $[a]$, with $a \in A$ non nilpotent, which by hypothesis are invertible in $A$, so also in $A/\mathfrak{N}$.
(c.$\Rightarrow$a.) Assume $A/\mathfrak{N}$ is a field, i.e. $(0)$ is the only prime ideal of $A/\mathfrak{N}$. Consider a prime $\mathfrak{p}$ of $A$. Then $\mathfrak{N} \subseteq \mathfrak{p}$ and so $\mathfrak{p}/\mathfrak{N}$ is a prime ideal of $A/\mathfrak{N}$, which implies that $\mathfrak{p} = \bigcap\limits_{\wp \subseteq A} \wp$ is the only prime ideal of $A$.

(5) This is a consequence of the following facts:

- For every prime ideal $\wp$ of $A$, the localization $\mathfrak{N}_\wp$ is the nilpotent radical of $A_\wp$.
- "Being zero for a $R$-module $M$ is a local property", that is,

$$M = 0 \iff M_\mathfrak{m} = 0 \text{ for every } \mathfrak{m} \text{ maximal ideal of } R.$$

The first fact can be checked elementwise, observing that the nilpotent radical of $A_\wp$ is

$$\bigcap_{\tilde{\mathfrak{p}} \subseteq A_\wp} \tilde{\mathfrak{p}} = \bigcap_{\substack{\mathfrak{p} \subseteq A \\ \mathfrak{p} \subseteq \wp}} \mathfrak{p} A_\wp = \bigcap_{\mathfrak{p} \subseteq A} \mathfrak{p} A_\wp,$$

since if $\mathfrak{p} \cap (A - \wp) \neq \emptyset$, then $\mathfrak{p} A_\wp = A_\wp$.

For the second one, suppose there is a $m \in M$, $m \neq 0$. Then $0 :_R m$ is a proper ideal of $R$, thus there exists a maximal ideal $\mathfrak{m}$ of $R$ with $\mathfrak{m} \supseteq 0 :_R m$. But by assumption $M_{\mathfrak{m}} = 0$, in particular $\frac{m}{1} = \frac{0}{1}$ in $M_{\mathfrak{m}}$. This means that there is an element $x \notin \mathfrak{m}$ so that $xm = 0$, i.e. $x \in 0 :_R m \subseteq \mathfrak{m}$, contradiction.

Being a zero-divisor is not a local property. Consider as a counterexample, $A = \mathbb{Z}/6\mathbb{Z}$. Clearly $A$ is not an integral domain, but its local rings are. The prime ideals of $A$ are $\wp_1 = 2\mathbb{Z}/6\mathbb{Z}$ and $\wp_2 = 3\mathbb{Z}/6\mathbb{Z}$. The maximal ideal of $A_{\wp_1}$ is

$$\begin{aligned}\wp_1 A_{\wp_1} &= \{\frac{a}{s} : a \in \wp_1,\ s \notin \wp_1\} \\ &= \{\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{0}{3}, \frac{2}{3}, \frac{4}{3}, \frac{0}{5}, \frac{2}{5}, \frac{4}{5}\}.\end{aligned}$$

You can easily check that every element in $\wp_1 A_{\wp_1}$ is equivalent to $\frac{0}{1}$. A local ring whose maximal ideal is zero is a field, hence an integral domain. Analogously for $\wp_2 A_{\wp_2}$.

6 Suppose that $\Sigma \neq \emptyset$. Every totally ordered subset $\Sigma_0$ of $\Sigma$ has a maximal element, namely $\bigcup_{I \in \Sigma_0} I$ (check this). Then by Zorn's lemma, $\Sigma$ has a maximal element $\mathfrak{a}$.

Suppose that $\mathfrak{a}$ is not a prime ideal. Then there are elements $x, y \notin \mathfrak{a}$, with

$$xy \in \mathfrak{a}.$$

The ideal $\mathfrak{a} + (x) \supset \mathfrak{a}$, so $\mathfrak{a} + (x) \notin \Sigma$ by the maximality of $\mathfrak{a}$. This means that $\mathfrak{a} + (x)$ is finitely generated, say

$$\mathfrak{a} + (x) = (x_1, \ldots, x_n, x)$$

with $x_1, \ldots, x_n \in \mathfrak{a}$.
Also, $\mathfrak{a} \subseteq \mathfrak{a} :_A x$ and the inclusion is strict ($y \in \mathfrak{a} :_A x$ and $y \notin \mathfrak{a}$). As above, we have that $\mathfrak{a} :_A x$ is finitely generated,

$$\mathfrak{a} :_A x = (y_1, \ldots, y_m).$$

Claim: $\mathfrak{a} = (x_1, \ldots, x_n, y_1 x, \ldots, y_m x)$.
($\supseteq$) Clear by definition of the elements $x_1, \ldots, x_n, y_1, \ldots, y_m$.
($\subseteq$) Let $a \in \mathfrak{a}$. Write $a + x \in \mathfrak{a} + (x)$ as

$$a + x = \sum a_i x_i + bx$$

with $a_i, b \in A$. Then

$$\begin{aligned}(1-b)x &= \sum a_i x_i - a \in \mathfrak{a} \\ \Longrightarrow 1 - b &\in \mathfrak{a} :_A x \\ \Longrightarrow 1 - b &= \sum c_i y_i\end{aligned}$$

for some $c_i \in A$. by multiplying both sides in the last equation by $x$ we obtain

$$a = \sum a_i x_i - \sum c_i y_i x.$$

Thus $\mathfrak{a}$ is finitely generated, a contradiction.