D-MATH
HS 2019
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Solutions 3

Commutative Algebra

(1) a. Let $x = \sum_{i=1}^{2} x^i e_i$ and $y = \sum_{i=1}^{2} y^i e_i$. Then $x \otimes y = \sum_{i,j} x^i y^j e_i \otimes e_j$. By applying ϕ one has

$$\begin{aligned}\phi(x \otimes y) &= \sum_{i,j} x^i y^j \phi(e_i \otimes e_j) \\ &= x^1 y^1 f_1 + x^1 y^2 f_2 + x^2 y^1 f_3 + x^2 y^2 f_4.\end{aligned}$$

Hence $\phi(x \otimes y) = af_1 + bf_2 + cf_3 + df_4$ if and only if

$$\begin{aligned} x^1 y^1 &= a \qquad & x^1 y^1 &= b \\ x^2 y^1 &= c \qquad & x^2 y^2 &= d, \end{aligned}$$

which implies $ad = bc$. On the other hand, if $ad = bc$, $a, b \neq 0$, pick $x^1 = 1$, $x^2 = c/a = d/b$, $y^1 = a$ and $y^2 = b$. Similarly for the other possibilities for a, b, c, d.

b. Let $u = u_1 \otimes u_2 : \mathbb{R}^2 \otimes \mathbb{R}^2 \to \mathbb{R}^2 \otimes \mathbb{R}^2$. Then the matrix of u is given by the components of $u(f_i)$ with respect to the basis $(f_i)_i$:

$$\begin{aligned} u(e_1 \otimes e_1) &= u_1(e_1) \otimes u_2(e_1) = (1, 0) \otimes (-1, 2) = -f_1 + 2f_2 \\ u(e_1 \otimes e_2) &= 4f_1 + 3f_2 \\ u(e_2 \otimes e_1) &= -2f_1 + 4f_2 - 3f_3 + 6f_4 \\ u(e_2 \otimes e_2) &= 8f_1 + 6f_2 + 12f_3 + 9f_4. \end{aligned}$$

Therefore

$$u = \begin{pmatrix} -1 & 4 & -2 & 8 \\ 2 & 3 & 4 & 6 \\ 0 & 0 & -3 & 12 \\ 0 & 0 & 6 & 9 \end{pmatrix}.$$

(2) a. Let $\phi : F \to E$ be linear of rank 1, i.e. $\dim(\operatorname{im} \phi) = 1$. Then $\operatorname{im} \phi = < \phi(\bar{f}) >_K$ with $\bar{f}$ any element not in the kernel of ϕ. For every $f \in F$, one has

$$\phi(f) = \eta_f \phi(\bar{f})$$

for an $\eta_f \in K$. Define $\lambda \in F'$ by $\lambda(f) = \eta_f$ for all $f \in F$. The map λ is linear since

$$\begin{aligned}\phi(\eta f + \mu g) &= \eta\phi(f) + \mu\phi(g)\\ &= (\eta\eta_f + \mu\mu_g)\phi(\bar{f})\end{aligned}$$

for $f, g \in F$, $\eta, \mu \in K$; so

$$\lambda(\eta f + \mu g) = \eta\eta_f + \mu\mu_g = \eta\lambda(f) + \mu\lambda(g).$$

One then has

$$\phi = u_{\lambda,\phi(\bar{f})}.$$

b. Consider the commutative diagram, given by the bilinearity of $\phi : (\lambda, x) \mapsto u_{\lambda,x}$,

$$\begin{tikzcd} F' \times E \arrow[rr, "\phi"] \arrow[dr, "\otimes"'] & & \mathrm{Hom}_K(F, E) \\ & F' \otimes E \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

If $\lambda(f)x = 0$ for all $f \in F$ then either $x = 0$ or $\lambda(f) = 0$ for all f, i.e. $\lambda = 0$. Hence $\lambda \otimes x = 0$, which means that $\tilde{\phi}$ is injective, and so it's an isomorphism since the two vector spaces have the same dimension.

c. Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ be basis of E and F, respectively $(n, m \geq 2)$. The element

$$f_1 \otimes e_1 + f_2 \otimes e_2$$

is not a pure tensor. If $f_1 \otimes e_1 + f_2 \otimes e_2 = f \otimes e$ for some $f \in F$, $e \in E$, then write $f = \sum \lambda^i f_i$ and $e = \sum \eta^i e_i$. It must be

$$\begin{cases} \lambda^1\eta^1 = 1 \\ \lambda^2\eta^2 = 1 \\ \lambda^1\eta^2 = 0 \\ \lambda^2\eta^1 = 0 \end{cases}$$

which has no solutions in K.

d. The map $<,>: E' \otimes E \to K$ is simply given as solutin of the universal problem, since the pairing $E' \times E \to K$ is bilinear. By composing:

$$\begin{tikzcd} \mathrm{End}_K(E) \arrow[r, "\phi"] \arrow[rr, bend right] & E' \otimes E \arrow[r, "<,>"] & K \end{tikzcd}$$

Let $(e_1, \ldots, e_n)$ be a basis of E. If $f \in \mathrm{End}_K(E)$, then $\mathrm{im}(f)$ is generated by $f(e_1), \ldots, f(e_n)$. Write, for $i = 1, \ldots, n$

$$\phi(e_i) = \sum_j a_{ij} e_j,$$

with $a_{ij} \in K$. Then for $e \in E$, there are $\alpha_e^j \in K$ so that

$$\begin{aligned} f(e) &= \sum_j \alpha_e^j \phi(e_j) \\ &= \sum_i (\sum_j \alpha_e^j a_{ji}) e_i \\ &= \sum_i u_{\lambda_i, e_i} \end{aligned}$$

with $(\lambda_i : e \mapsto \sum_j \alpha_e^j a_{ji}) \in E'$. Therefore

$$f \overset{\phi}{\longmapsto} \sum_i \lambda_i \otimes e_i \overset{<,>}{\longmapsto} \sum_i \lambda_i(e_i) = \sum_i a_{ii},$$

so

$$<,> \circ \phi = \mathrm{trace}(\cdot).$$

e. Let $L^2(E, F; K)$ be the space of K-bilinear maps $E \times F \to K$. Consider the composition of isomorphisms

$$\begin{aligned} E' \otimes_K F' &\longrightarrow \mathrm{Hom}_K(E, F') = \mathrm{Hom}_K(E, \mathrm{Hom}_K(F, K)) \\ &\longrightarrow L^2(E, F; K) \longrightarrow \mathrm{Hom}_K(E \otimes_K F, K) = (E \otimes F)' \end{aligned}$$

given by

$$\lambda \otimes \mu \longmapsto u_{\lambda,\mu} \longmapsto ((e, f) \mapsto u_{\lambda,\mu}(e)(f)) \longmapsto (e \otimes f \mapsto u_{\lambda,\mu}(e)(f)).$$

3 For any A-module P and for any bilinear map $M \times N \xrightarrow{f} P$ there is a linear $\tilde{f}$ so that the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[rr, "f"] \arrow[dr, "\beta"'] & & P \\ & M \otimes_A N \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Let P be the A-submodule of $M \otimes_A N$ generated by $(\beta(m, n))_{(m,n) \in M \otimes_A N}$ and $f(m, n) = \beta(m, n)$. Then there is a unique g so that $g \circ \beta = \beta$. In particular g is surjective on P, and so also injective (the inverse is the natural inclusion $P \hookrightarrow M \otimes_A N$). Hence $M \otimes_A N \simeq P$.

4 The map b is bilinear, so again $\tilde{b} : L(X_1) \otimes_{\mathbb{C}} L(X_2) \longrightarrow L(X_1 \times X_2)$ is the solution of the universal problem. It's easy to check that $\tilde{b}$ is injective. Note also that $\dim L(X_i) = |X_i|$ (show for example that if $X_i = \{x_1, \dots, x_n\}$, $(f_j)_{j=1,\dots,|X_i|}$ with $f_j(x) = \begin{cases} 1 & \text{if } x = x_j \\ 0 & \text{otherwise} \end{cases}$ is a basis of $L(X_i)$). Thus

$$\dim_{\mathbb{C}}(L(X_1) \otimes_{\mathbb{C}} L(X_2)) = \dim_{\mathbb{C}}(X_1 \times X_2),$$

so $\tilde{b}$ is an isomorphism.

5 b. The map

$$\begin{aligned} M \times N \times P &\longrightarrow M \otimes (N \otimes P) \\ (m, n, p) &\longmapsto m \otimes (n \otimes p) \end{aligned}$$

is 3-linear, so fixing $p \in P$ we have a bilinear

$$\begin{aligned} \phi_p : M \times N &\longrightarrow M \otimes (N \otimes P) \\ (m, n) &\longmapsto m \otimes (n \otimes p) \end{aligned}$$

which induces a linear

$$\begin{aligned} \tilde{\phi}_p : M \otimes N &\longrightarrow M \otimes (N \otimes P) \\ m \otimes n &\longmapsto m \otimes (n \otimes p). \end{aligned}$$

Now consider the bilinear

$$\begin{aligned} (M \otimes N) \times P &\longrightarrow M \otimes (N \otimes P) \\ (m \otimes n, p) &\longmapsto \tilde{\phi}_p(m \otimes n); \end{aligned}$$

it iduces a linear map

$$\begin{aligned} f : (M \otimes N) \otimes P &\longrightarrow M \otimes (N \otimes P) \\ (m \otimes n) \otimes p &\longmapsto \tilde{\phi}_p(m \otimes n) = m \otimes (n \otimes p). \end{aligned}$$

Similarily ther is a linear map

$$\begin{aligned} g : M \otimes (N \otimes P) &\longrightarrow (M \otimes N) \otimes P \\ m \otimes (n \otimes p) &\longmapsto (m \otimes n) \otimes p. \end{aligned}$$

One has $f \circ g(m \otimes (n \otimes p)) = m \otimes (n \otimes p)$ and $g \circ f((m \otimes n) \otimes p) = (m \otimes n) \otimes p$. Since the pure tensors span the two modules, these identities extend by linearity to show that f and g are inverse functions.

a. Use the universal property for $M \otimes N$ and $N \otimes M$ and show that the two linear maps are inverse.

6 In $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ one has

$$2 \otimes [1] = 1 \otimes 2[1] = 1 \otimes [0] = 0.$$

In general, for an A-module M we have a canonical isomorphism of A-modules

$$\begin{aligned} \phi : M \otimes_A A/I &\longrightarrow M/IM \\ m \otimes [a] &\longmapsto [am]. \end{aligned}$$

So in our case

$$2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq 2\mathbb{Z}/(2)2\mathbb{Z} \simeq 2\mathbb{Z}/4\mathbb{Z},$$

and via ϕ, $2 \otimes [1]$ is sent to $[2] \in 2\mathbb{Z}/4\mathbb{Z}$, which is not 0, so $2 \otimes [1]$ is not 0 in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.