D-MATH
HS 2019
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Solutions 4

Commutative Algebra

1. a. The map $f \circ d : B \to M'$ is an A-derivation, since, using the B-linearity of f,

$$\begin{aligned} f(d(bb')) &= f(bdb' + b'db) = bf(db') + b'f(db) \\ f(d(s(a))) &= f(0) = 0 \end{aligned}$$

for all $b, b' \in B$, $a \in A$.

b. Consider the free B-module

$$\Omega := B^{dB} = \bigoplus_{db \in dB} B$$

generated by the set

$$dB := \{db : b \in B\}.$$

Define

$$\Omega_{B/A} := \Omega/\Omega'$$

where Ω is the B-submodule generated by the elements

$$\begin{aligned} &d(bb') - bdb' - b'db \\ &d(b + b') - db - db' \\ &d(s(a)) \end{aligned}$$

for $b, b' \in B$, $a \in A$. Define

$$\begin{aligned} d_u : B &\longrightarrow \Omega_{B/A} \\ b &\longmapsto [db] \end{aligned}$$

and

$$\begin{aligned} f : \Omega_{B/A} &\longrightarrow M \\ [db] &\longmapsto db. \end{aligned}$$

Then f is well-defined and has the desired properties. For the uniqueness, note that the universal derivation d_u is surjective, so f is determined by $f \circ d_u = d$.

To conclude, $\Omega_{B/A}$ is unique up to B-isomorphism: consider $M = \Omega_{B/A}$, $f = d_u$ and let $\Omega'_{B/A}$, $d'_u : B \to \Omega'_{B/A}$ another solution of the universal problem.

$$\begin{tikzcd}
B \arrow[r, "d_u"] \arrow[dr, "d'_u"'] & \Omega_{B/A} \arrow[d, bend left, "f"] \\
& \Omega'_{B/A} \arrow[u, bend left, "f'"]
\end{tikzcd}$$

Let f' be the B-linear map such that $f' \circ d'_u = d_u$. By the above property of $\Omega_{B/A}$, there is also a B-linear f so that $f \circ d_u = d'_u$. Therefore

$$f' \circ f \circ d_u = d_u,$$

which implies $f' \circ f = 1$ by the surjectivity of d_u.

2 The map $f : B \otimes_A N \to B \otimes_A M$ is the uniquely defined B-linear map so that the following square commutes:

$$\begin{tikzcd}
N \arrow[r, hook, "\iota"] \arrow[d, "1 \otimes \mathrm{id}_N"'] & M \arrow[d, "1 \otimes \mathrm{id}_M"] \\
B \otimes_A N \arrow[r, "f"] & B \otimes_A M
\end{tikzcd}$$

Let $A = M = \mathbb{Z}$, $B = \mathbb{Z}/2\mathbb{Z}$, $N = 2\mathbb{Z}$, then f is given by

$$\begin{aligned} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} 2\mathbb{Z} &\longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \\ [1] \otimes 2 &\longmapsto [1] \otimes 2. \end{aligned}$$

By exercise number 7 of the sheet 3, $[1] \otimes 2$ is not 0 in $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} 2\mathbb{Z}$, but it's 0 in $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$, so f is not injective.

3 For every B-module L, a A-linear map $f : M \to L$ induces a B-linear map:

$$\begin{aligned} f' : B \otimes_A M &\longrightarrow L \\ b \otimes m &\longmapsto bf(m). \end{aligned}$$

We have a natural A-linear map

$$\begin{aligned} \mathrm{Hom}_A(M, M') &\longrightarrow \mathrm{Hom}_B(B \otimes_A M, B \otimes_A M') \\ \alpha &\longmapsto \mathrm{id}_B \otimes \alpha. \end{aligned}$$

Since on the right-hand side we have a B-module, we get an induced B-morphism

$$\Psi : B \otimes_A \mathrm{Hom}_A(M, M') \longrightarrow \mathrm{Hom}_B(B \otimes_A M, B \otimes_A M')$$

given by

$$\Psi(b \otimes \alpha) = b(\mathrm{id}_B \otimes \alpha)$$

for $b \in B$, $\alpha \in \mathrm{Hom}_A(M, M')$.

Let $A = M' = \mathbb{Z}$ $B = M = \mathbb{Z}/2\mathbb{Z}$, then

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\Psi} \mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}).$$

it's easy to check that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$, so the first module is 0. The second one is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so Ψ is not an isomorphism.

The map $\Phi : (B \otimes_A M) \otimes_B (B \otimes_A M') \to B \otimes_A (M \otimes_A M')$ is induced by the B-bilinear map

$$\begin{aligned} (B \otimes_A M) \times (B \otimes_A M') &\longrightarrow B \otimes_A (M \otimes_A M') \\ (b \otimes m, b' \otimes m') &\longmapsto bb' \otimes (m \otimes m'). \end{aligned}$$

The inverse of Φ is given by

$$b \otimes (m \otimes m') \longmapsto (b \otimes m) \otimes (1 \otimes m') = (1 \otimes m) \otimes (b \otimes m')$$

for all $b \in B$, $m \in M$, $m' \in M'$.

4

a. As in the previuos case, F is induced by the corresponding A-bilinear map, and it's given by

$$F(f_1 \otimes f_2) = (m_1 \otimes m_2 \mapsto f_1(m_1) \otimes f_2(m_2))$$

for every $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_A(M_2, N_2)$, $m_1 \in M_1$ and $m_2 \in M_2$.

b. Since the vector spaces $\mathrm{Hom}_K(M_1, N_1) \otimes_K \mathrm{Hom}_K(M_2, N_2)$ and $\mathrm{Hom}_K(M_1 \otimes_K M_2, N_1 \otimes_K N_2)$ have the same dimension, it's enough to check the injectivity of F. Let $f_1 \in \mathrm{Hom}_K(M_1, N_1)$, $f_2 \in \mathrm{Hom}_K(M_2, N_2)$ such that $f_1(m_1) \otimes f_2(m_2) = 0$ for all $m_1 \in M_1$ and $m_2 \in M_2$. Observe that if $f_i(m_i) \neq 0$ $(i = 1, 2)$, then $f_i(m_i)$ is part of a basis of N_i, so $f_1(m_1) \otimes f_2(m_2)$ is part of a basis of $N_1 \otimes_K N_2$, and it cannot be zero. On the other hand if $f_1(m_1) = 0$ or $f_2(m_2) = 0$, then the tensor product is 0. Since this holds for all m_1, m_2, we conclude that $f_1 = 0$ or $f_2 = 0$, so $f_1 \otimes f_2 = 0$.

c. To abbreviate, we'll denote an element of $\mathbb{Z}/4\mathbb{Z}$ without the symbol of class and by $[\cdot]$ an element in the quotient $(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})$. In general, for an A-module M and an ideal $I \subseteq A$, we have

$$\mathrm{Hom}_A(A/I, M) \simeq 0 :_M I,$$

and by the A-freeness of A,

$$\mathrm{Hom}_A(A, M) \simeq M.$$

Hence we have an ismorphism

$$\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}),\mathbb{Z}/4\mathbb{Z})\otimes_{\mathbb{Z}/4\mathbb{Z}}\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z},(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})) \\ \simeq 2\mathbb{Z}/4\mathbb{Z}\otimes_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})$$

given by $\phi\otimes\psi \mapsto \phi([1])\otimes\psi(1)$ for all $\phi \in \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}),\mathbb{Z}/4\mathbb{Z})$, $\psi \in \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z},(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}))$. By a similar argument,

$$\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \\ \otimes_{\mathbb{Z}/4\mathbb{Z}}\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})\otimes_{\mathbb{Z}/4\mathbb{Z}}\mathbb{Z}/4\mathbb{Z},\mathbb{Z}/4\mathbb{Z}\otimes_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})) \\ \simeq \mathbb{Z}/4\mathbb{Z}\otimes_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \\ \alpha \mapsto \alpha([1]\otimes 1).$$

Moreover,

$$2\mathbb{Z}/4\mathbb{Z}\otimes_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \simeq 2\mathbb{Z}/4\mathbb{Z} \\ 2\otimes[1]\mapsto 2.$$

The corresponding map $\tilde{F}$ induced by F sends 2 to $2\otimes[1]$, which is 0 in $\mathbb{Z}/4\mathbb{Z}\otimes_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})$. Clearly $1\otimes[1]\notin \mathrm{im}\,\tilde{F}$.

d. As an $\mathbb{R}$-vector space, $\mathbb{C}\simeq\mathbb{R}\oplus\mathbb{R}$; since the tensor product commutes with finite direct sums we have isomorphisms of $\mathbb{R}$-vector spaces

$$\begin{aligned}\mathbb{C}^n\otimes_{\mathbb{R}}\mathbb{C}^m &\simeq \mathbb{R}^{\oplus 2n}\otimes_{\mathbb{R}}\mathbb{C}^m \\ &\simeq (\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}^m)^{\oplus 2n} \\ &\simeq (\mathbb{C}^m)^{2n} \\ &\simeq \mathbb{R}^{4mn}.\end{aligned}$$

5 Let V be of dimension n over K and W of dimension m over K. Choose basis $\mathscr{E}=\{v_1,\dots,v_n\}$ and $\mathscr{F}=\{w_1,\dots,w_m\}$ of V and W, respectively. Denote by

$$\begin{aligned}A &= A^{\mathscr{E}}(\phi)=(a_{ij}),\\ B &= B^{\mathscr{F}}(\phi)=(b_{ij}),\\ C &= C^{\mathscr{E}\otimes\mathscr{F}}(\phi\otimes\psi)=(c_{ij})\end{aligned}$$

the matrices associated to $\phi,\ \psi,\ \phi\otimes\psi$ with respect to the basis indicated, where $\mathscr{E}\otimes\mathscr{F}=\{v_i\otimes w_j\}_{i,j}$ (choose an order for the element of this basis of $V\otimes_K W$). For all $i=1,\dots,n$, $j=1,\dots,m$ one has

$$\begin{aligned}\phi\otimes\psi(v_i\otimes w_j) &= \phi(v_i)\otimes\psi(w_j)\\ &= \sum_k a_{ki}v_k\otimes\sum_h b_{hj}w_h\\ &= \sum_{k,h} a_{ki}b_{hj}(v_k\otimes w_h),\end{aligned}$$

which means that C is the matrix given by the Kronecker product

$$C = A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B. \end{pmatrix}$$

a.
$$\operatorname{Tr} C = a_{11} \operatorname{Tr} B + \cdots + a_{nn} \operatorname{Tr} B = \operatorname{Tr} A \operatorname{Tr} B.$$

b. The image of $\phi \otimes \psi$ is generated by $\{\phi(v_i) \otimes \psi(w_j)\}_{i,j}$. If $\phi(v_{i_1}), \ldots, \phi(v_{i_k})$ $(1 \leq i_1 < \cdots < i_k \leq n)$ and $\psi(w_{j_1}), \ldots, \psi(w_{j_h})$ $(1 \leq j_1 < \cdots < j_h \leq m)$ are basis of $\operatorname{im} \phi$, $\operatorname{im} \psi$, respectively, then $\{\phi(v_{i_{l_1}}) \otimes \psi(w_{j_{l_2}})\}_{l_1, l_2}$ is a basis of $\operatorname{im}(\phi \otimes \psi) = \operatorname{im} \phi \otimes \operatorname{im} \psi$.

c. To prove the formula for the determinant of the Kronecker product of matrices, consider the map $\phi \otimes \mathrm{id}_W$ and choose the following order for the basis

$$\mathscr{B} := \mathscr{E} \otimes \mathscr{F} = \{v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_1 \otimes w_m, v_2 \otimes w_1, \ldots, v_n \otimes w_m\}.$$

Then

$$\phi \otimes \mathrm{id}_W (v_i \otimes w_a) = \phi(v_i) \otimes w_a = \sum_{j,b} a_{ji} \delta_{ab} v_j \otimes w_b.$$

This implies that the matrix D associated to the basis $\mathscr{B}$ is the block-matrix

$$D = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$$

The determinant is the product of the determinants of the blocks, i.e.

$$\det D = (\det A)^m.$$

Analogously,

$$\det(\mathrm{id}_V \otimes \psi) = (\det B)^n.$$

The desired formula is obtained by observing that

$$\phi \otimes \psi = (\phi \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes \psi).$$

d. In general, using the definition, for any $\phi, \phi' \in \operatorname{Hom}_K(V, V)$ and $\psi, \psi' \in \operatorname{Hom}_K(W, W)$, one has

$$(\phi \otimes \psi)(\phi' \otimes \psi') = \phi\phi' \otimes \psi\psi'.$$

Assume ψ and ψ diagonalizable; then there are invertible matrices U, L and diagonal matrices Δ_A, Δ_B so that

$$A = U^{-1} \Delta_A U,$$

$$B = L^{-1}\Delta_B L.$$

It follows that

$$\begin{aligned} A \otimes B &= (U^{-1}\Delta_A U) \otimes (L^{-1}\Delta_B L) \\ &= (U^{-1} \otimes L^{-1})(\Delta_A U \otimes \Delta_B L) \\ &= (U^{-1} \otimes L^{-1})(\Delta_A \otimes \Delta_B)(U \otimes L). \end{aligned}$$

Conclude observing that $\Delta_A \otimes \Delta_B$ is diagonal and $(U \otimes L)^{-1} = U^{-1} \otimes L^{-1}$.

e. Let $s > 0$ so that $\phi^s = 0$. Then by the above it follows that

$$(A \otimes B)^s = A^s \otimes B^s = 0.$$