D-MATH
HS 2019
Prof. E. Kowalski

Solutions 5

Commutative Algebra

(1) a. For all $b \in B$ the map

$$\begin{aligned} \varphi_b : M \times N &\longrightarrow M \otimes_A N \\ (m, n) &\longmapsto bm \otimes n \end{aligned}$$

is A-bilinear, so there is a unique A-linear map

$$\varphi_b : M \otimes_A N \longrightarrow M \otimes_A N$$

sending $m \otimes n$ to $bm \otimes n$ for all $b \in B,\ m \in M,\ n \in N$. It's easy to check that φ_b is also B-linear, so the ring morphism

$$\begin{aligned} B &\longrightarrow \mathrm{End}_{\mathbb{Z}}(M \otimes_A N) \\ b &\longmapsto \varphi_b \end{aligned}$$

defines the unique B-module structure with the desired property.

b. By the universal property, there is a unique A-linear map

$$\begin{aligned} \varphi : M \times N &\longrightarrow M \otimes_B (B \otimes_A N) \\ m \otimes n &\longmapsto m \otimes (1 \otimes n). \end{aligned}$$

For all $b \in B,\ m \in M,\ n \in N$,

$$\begin{aligned} \varphi(b(m \otimes n)) &= \varphi(bm \otimes n) \\ &= bm \otimes (1 \otimes n) \\ &= b(m \otimes (1 \otimes n)) = b\varphi(m \otimes n), \end{aligned}$$

so φ is B-linear. The inverse is the B-linear map given by the B-bilinearity of

$$\begin{aligned} M \times (B \otimes_A N) &\longrightarrow M \otimes_A N \\ (m, b \otimes n) &\longmapsto bm \otimes n. \end{aligned}$$

(2) Fix an A-isomorphism $\varphi : M \to N$. Pick a maximal ideal $\mathfrak{m}$ of $A \neq 0$. Then $k := A/\mathfrak{m}$ is an A-module, and the map

$$\begin{aligned} \mathrm{id}_k \otimes \varphi : k \otimes_A M &\longrightarrow k \otimes_A N \\ \lambda \otimes m &\longmapsto \lambda \otimes \varphi(m) \end{aligned}$$

is k-linear. If $(e_i)_{i=1,\ldots,m}$ is a basis of M over A, then $([e_i])_i$ is a basis of $M/\mathfrak{m}M \sim k \otimes_A M$ over k, thus $\dim_k M/\mathfrak{m}M = m$ and analogously $\dim_k N/\mathfrak{m}N = n$.

To conclude, note that the inverse of $\mathrm{id}_k \otimes \varphi$ is $\mathrm{id}_k \otimes \varphi^{-1}$, so $\mathrm{id}_k \otimes \varphi$ is an isomorphism of k-vector spaces, which implies $m = n$.

(3) a. Denote by

$$\begin{aligned} \varphi_1 : K &\longrightarrow \mathrm{End}_{\mathbb{Z}}(E \otimes_A F) \\ \lambda &\longmapsto (e \otimes f \overset{\varphi_\lambda}{\mapsto} \lambda e \otimes f) \end{aligned}$$

and

$$\begin{aligned} \varphi_2 : K &\longrightarrow \mathrm{End}_{\mathbb{Z}}(E \otimes_A F) \\ \lambda &\longmapsto (e \otimes f \overset{\varphi^\lambda}{\mapsto} e \otimes \lambda f) \end{aligned}$$

the two K-vector space stuctures on $E \otimes_A F$.
Pick basis $(e_i)_i$ of E and $(f_j)_j$ of F. Write and element $e \otimes f \in (E \otimes_A F)^{\varphi_1}$ as

$$e \otimes f = \sum \frac{a_{ij}}{b_{ij}} e_i \otimes f_j,$$

with $a_{ij}, b_{ij} \in A,\ b_{ij} \neq 0$. Define

$$\alpha : E \otimes_A F \longrightarrow E \otimes_A F$$

by

$$\alpha(e \otimes f) = \sum e_i \otimes \frac{a_{ij}}{b_{ij}} f_j.$$

Then clearly α is an isomorphism of k-vector spaces, and for all $\lambda \in K$, the following diagram commutes:

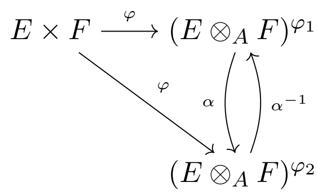
$$\begin{array}{ccc} E \otimes_A F & \xrightarrow{\alpha} & E \otimes_A F \\ \varphi_\lambda \downarrow & & \downarrow \varphi^\lambda \\ E \otimes_A F & \xrightarrow{\alpha} & E \otimes_A F \end{array}$$

i.e. $\varphi^\lambda \circ \alpha = \alpha \circ \varphi_\lambda$ for all $\lambda \in K$, which means that φ_1 and φ_2 define the same k-vector space structure.

b. Consider the map

$$\begin{aligned} \varphi : E \times F &\longrightarrow E \otimes_A F \\ (e, f) &\longmapsto e \otimes f \end{aligned}$$

and the diagram



Observe that φ to $(E \otimes_A F)^{\varphi_1}$ is linear of the first component, but $\alpha \circ \varphi$ is linear on the second component, and vice versa. Since in $E \otimes_K F$, $\lambda(e \otimes f) = \lambda e \otimes f = e \otimes \lambda f$ for all $\lambda \in K,\ e \in E,\ f \in F$, there is an induced K-linear map

$$E \otimes_K F \longrightarrow E \otimes_A F.$$

The inverse is the induced A-linear map by

$$\begin{aligned} E \times F &\longrightarrow E \otimes_K F \\ (e, f) &\longmapsto e \otimes f, \end{aligned}$$

which turns out to be K-linear, too.

(4) Define

$$\begin{aligned} \Phi : \mathrm{Hom}_A(M \otimes_A N, L) &\longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, L)) \\ \phi &\longmapsto (m \mapsto (n \mapsto \phi(m \otimes n))) \end{aligned}$$

and

$$\begin{aligned} \Psi : \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, L)) &\longrightarrow \mathrm{Hom}_A(M \otimes_A N, L) \\ \psi &\longmapsto (m \otimes n \mapsto \psi(m)(n)). \end{aligned}$$

Then Φ and Ψ are morphisms of A-modules, mutually inverse:

$$\Psi \circ \Phi(\phi) = (m \otimes n \mapsto \Phi(\phi)(m)(n)) = (m \otimes n \mapsto \phi(m \otimes n)) = \phi$$
$$\Phi \circ \Psi(\psi) = (m \mapsto (n \mapsto \Psi(\psi)(m \otimes n))) = (m \mapsto (n \mapsto \psi(m)(n))) = \psi.$$

(5) a. Let $(e_i)_{i \in I}$ be a basis of F, free A-module. Then by the universal property of free modules, a choice of a set $\{m_i : i \in I\}$ of M defines a unique A-linear morphism $h \in \mathrm{Hom}_A(F, M)$ by $h(e_i) = m_i$ for all $i \in I$. Hence choose $m_i \in M$ so that $f(m_i) = g(e_i)$. Then

$$f(h(e_i)) = f(m_i) = g(e_i)\ \forall i \in I,$$

which implies that $f \circ h = g$.

b. Pick a system of generators $(m_i)_{i\in I}$ of M (at worst take all the elements of M). Let X be the free A-module A^I with basis $(e_i)_{i\in I}$, and

$$\begin{aligned} \varepsilon : X &\longrightarrow M \\ e_i &\longmapsto m_i. \end{aligned}$$

c. Note that $\mathbb{Q}$ as $\mathbb{Z}$-module is generated by $\{\frac{1}{n} : n \in \mathbb{N}_{>0}\}$;

$$\begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Q}}} \\ \bigoplus_{n>0}\mathbb{Z} & \xrightarrow{f} & \mathbb{Q} \end{array}$$

Using the universal property, let

$$\begin{aligned} f : M &\longrightarrow \mathbb{Q} \\ e_n &\longmapsto \frac{1}{n}. \end{aligned}$$

If there's a $\mathbb{Z}$-linear map $h : \mathbb{Q} \to M$, then for all $\frac{a}{b} \in \mathbb{Q}$ and for all $n > 0$,

$$h\Big(\frac{a}{b}\Big) = nh\Big(\frac{a}{nb}\Big).$$

But $h\big(\frac{a}{b}\big)$ is a "vector" of integers, so the only possibility is $h\big(\frac{a}{b}\big) = 0$ for all $\frac{a}{b} \in \mathbb{Q}$, so there is not a map h so that the above diagram commutes.