D-MATH
HS 2019
Prof. E. Kowalski

# Solutions 6

Commutative Algebra

**1** One example is the exact sequence induced by the multiplication by 2 in $\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

If there exist a linear map $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ such that $\pi \circ f = 1$, then

$$2f(1) = f(2) = 0,$$

so $f$ is zero, a contadiction.

Another interesting example is the following exact sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{e^{2\pi i \cdot}} \mathbb{C}^{\times} \longrightarrow 0.$$

If the sequence is split, then there is a linear $f : \mathbb{C} \to \mathbb{Z}$ such that

$$f(z) = \begin{cases} z & \text{if } z \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

But this implies that

$$1 = f(1) = 2f(1/2) = 0.$$

**2**

a. An equivalent definition of $E$ divisible is that for every ideal $(a) \subseteq A$, $a$ non zero-divisor, every map $(a) \xrightarrow{s} E$ extends to $A \xrightarrow{h} E$ (observe that since $a$ is a non zero-divisor, $s(a)$ can be any element of $E$). So consider $N = (a)$, $M = A$ in the diagram.

b. We use the following theorem (Baer criterion): $E$ is injective if and only if for every ideal $I$ of $A$ and for every $A$-linear map $s : I \to E$ there is an $h$ such that the diagram commutes

$$\begin{tikzcd} & E \\ A \arrow[ur, "h"] & I \arrow[l, hook', "f"'] \arrow[u, "s"'] \end{tikzcd}$$

For a proof see Proposition 6.4.2 of Antoine Chambert-Loir's book. So the implication ($\Leftarrow$) follows, since the non zero ideals of $A$ are principal, generated by a non zero-divisor.

c. $\mathbb{Q}$ as $\mathbb{Z}$-module is injective, since divisible. The $\mathbb{Z}$-module $\mathbb{Z}$ is not injective, since it's clearly non divisible.

**3** The main point is that over a PID, a submodule of a free module is free.

a. Let $n$ be the number of generators of $M$ over $A$. Then there is a presentation of $M$

$$A^n \overset{\varepsilon}{\twoheadrightarrow} M.$$

The kernel of $\varepsilon$ is a free module, say of dimension $m$, so its projective presentation with a projective module is $A^m \to \ker \varepsilon$, which is injective.

b. If $M$ is free of rank $n$, then $M \simeq A^n$, so we clearly have a sequence $0 \to A^n \to M \to 0$. On the other hand, if in the sequence in a. the presentation $\varepsilon$ is injective, then $M \simeq A^n$, so $M$ is free.

c. The module $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$ has 3 generators over $\mathbb{Z}$, so we start with the presentation

$$\mathbb{Z}^3 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where $\varepsilon$ is defined on the basis $e_1, e_2 e_3$ of $\mathbb{Z}^3$ as

$$\begin{aligned} e_1 &\longmapsto (1 \mod 2, 0, 0) \\ e_1 &\longmapsto (0, 1 \mod 4, 0) \\ e_1 &\longmapsto (0, 0, 1). \end{aligned}$$

The kernel of $\varepsilon$ is $2\mathbb{Z} \oplus 4\mathbb{Z} \oplus \{0\}$, we present it with $\mathbb{Z}^2$ and consider the composition $d_1$

$$\begin{tikzcd} \mathbb{Z}^2 \arrow[r, "d_1"] \arrow[d, two heads] & \mathbb{Z}^3 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\ \ker \varepsilon \arrow[ur, hook] & & & \end{tikzcd}$$

The map $d_1$ sends $e_1$ to $(2, 0, 0)$ and $e_2$ to $(0, 4, 0)$, it is injective, so we have the free resolution

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}^3 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

d. We start with the projection

$$\mathbb{Q}[X] \xrightarrow{\pi} M \longrightarrow 0.$$

The kernel is $(X^2+1)$, which is a free $\mathbb{Q}[X]$-module of rank one, so by proceeding as before we have the free resolution

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q}[X] \arrow[r, "d_1"] \arrow[d, two heads] & \mathbb{Q}[X] \arrow[r, "\pi"] & M \arrow[r] & 0 \\ & (X^2+1) \arrow[ur, hook] & & & \end{tikzcd}$$

**4** a. Let $n_3 \in N_3$. Then
$\exists m_4 \in M_4$ with $h_4(m_4) = g_3(n_3)$
$\Rightarrow g_4 g_3(n_3) = 0 = g_4 h_4(m_4) = h_5 f_4(m_4)$
$\Rightarrow f_4(m_4) = 0$
$\Rightarrow \exists m_3 \in M_3$ with $f_3(m_3) = m_4$
$\Rightarrow g_3(n_3 - h_3(m_3)) = h_4(m_4) - h_4 f_3(m_3) = h_4(m_4) - h_4(m_4) = 0$
$\Rightarrow \exists n_2 \in N_2$ with $g_2(n_2) = n_3 - h_3(m_3)$
$\Rightarrow \exists m_2 \in M_2$ with $h_2(m_2) = n_2$.
Then

$$\begin{aligned} h_3(f_2(m_2) - m_3) &= h_3 f_2(m_2) + h_3(m_3) \\ &= g_2 h_2(m_2) + h_3(m_3) \\ &= g_2(n_2) + h_3(m_3) \\ &= n_3 - h_3(m_3) + h_3(m_3) \\ &= n_3. \end{aligned}$$

b. Let $m_3 \in M_3$ with $h_3(m_3) = 0$. Then
$h_4 f_3(m_3) = g_3 h_3(m_3) = 0$
$\Rightarrow f_3(m_3) = 0$
$\Rightarrow \exists m_2 \in M_2$ with $f_2(m_2) = m_3$
$\Rightarrow g_2 h_2(m_2) = h_3 f_2(m_2) = h_3(m_3) = 0$
$\Rightarrow g_3(n_3 - h_3(m_3)) = h_4(m_4) - h_4 f_3(m_3) = h_4(m_4) - h_4(m_4) = 0$
$\Rightarrow \exists n_1 \in N_1$ with $g_1(n_1) = h_2(m_2)$
$\Rightarrow \exists m_1 \in M_1$ with $h_1(m_1) = n_1$
$\Rightarrow g_1 h_1(m_1) = g_1(n_1) = h_2 f_1(m_1) = h_2(m_2)$
$\Rightarrow f_1(m_1) = m_2$
$\Rightarrow f_2(m_2) = f_2 f_1(m_1) = 0 = m_3$.
Thus $h_3$ is injective.

c. Trivial from a. and b..

**5** Proceed by induction over $n$.
$n = 1$: Trivial, since in this case $E_1 = 0$.
$n > 1$: The exact sequence

$$0 \longrightarrow E_{n+1} \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow 0$$

induces the following exact sequence of length $n$

$$0 \longrightarrow E_n/E_{n+1} \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow 0.$$

By induction, we have that

$$\sum_{i=1}^{n-1}(-1)^i \dim E_i + (-1)^n \dim(E_n/E_{n+1}) = 0.$$

But $\dim(E_n/E_{n+1}) = \dim E_n - \dim E_{n+1}$, so the above is the desired formula for sequences of length $n+1$.

**6** Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be the short exact sequence. Let $M' =< x_1, \dots, x_n >_A$ and $M'' =< g(y_1), \dots, g(y_m) >_A$ for some $y_i \in M$. If $x \in M$, then there exist $b_i \in A$ such that

$$\begin{aligned} g(x) &= \sum b_i g(y_i) = g(\sum b_i y_i) \\ \implies x &- \sum b_i y_i \in \ker g = \operatorname{im} f, \end{aligned}$$

hence there are scalars $a_i \in A$ so that

$$\begin{aligned} x - \sum b_i y_i &= \sum a_i f(x_i) = g(\sum b_i y_i) \\ \implies x &= \sum b_i y_i + \sum a_i f(x_i), \end{aligned}$$

so $M$ is generated by $f(x_1), \dots, f(x_n), y_1, \dots, y_m$.

**7** The projection map

$$A \overset{\varepsilon}{\twoheadrightarrow} A/(X^m)$$

has kernel $(X^m)$. We present it with the multiplication by $X^m$ map. The composition $d_1$ with the natural embedding has kernel

$$\{f \in A : X^m f \in (X^n)\} = (X^{n-m}).$$

The inclusion $(\supseteq)$ is clear. For $(\subseteq)$ take a representative of $f$ in $\mathbb{Q}[X]$. Then there is a $g \in \mathbb{Q}[X]$ so that $X^m f = gX^n$ in $\mathbb{Q}[X]$. This implies $f = gX^{n-m}$, since $\mathbb{Q}[X]$ is an integral domain. Again, the presentation of $(X^{n-m})$ is the multiplication by $X^{n-m}$ map, whose composition with the embeddig has kernel $(X^m)$, and we can continue this process infinitely many times, obtaining an infinite free resolution

$$\begin{tikzcd}
\cdots A \arrow[r, "d_3"] \arrow[dr, two heads] & A \arrow[r, "d_2"] \arrow[dr, two heads] & A \arrow[r, "d_1"] \arrow[dr, two heads] & A \arrow[r, "\varepsilon"] & A/(X^m) \arrow[r] & 0 \\
& (X^m) \arrow[u, hook] & (X^{n-m}) \arrow[u, hook] & (X^m) \arrow[u, hook] & &
\end{tikzcd}$$