D-MATH
HS 2019
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Solutions 7

Commutative Algebra

1 We consider the case $n = 2$, the general case follows by induction. First, note that all the ideals of $A_1 \times A_2$ are of the form $I \times J$, with I, J ideals of A_1 and A_2, respectively. Moreover, it's easy to show that the prime ideals of $A_1 \times A_2$ are of the form $A \times \wp_2$, $\wp_1 \times A_2$ with $\wp_1 \subseteq A_1$, $\wp_2 \subseteq A_2$ prime ideals. Therefore, any chain of prime ideals in $A_1 \times A_2$ arises from either a chain of primes in A_1 or a chain of primes in A_2. The longest chain must come from the longest chain in A_1 or A_2.

2 Let $M = \oplus_{i \in I} M_i$. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow K.$$

Tensoring with M gives

$$N \otimes M \longrightarrow K \otimes M$$

or

$$\oplus_{i \in I}(N \otimes M_i) \longrightarrow \oplus_{i \in I}(K \otimes M_i).$$

Each of the coordinate maps is injective, hence the above is injective. In particular, $\_ \otimes_A A$ is exact.

3

a. There are many ways to solve it; one of them is to show that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. Observe that in $\mathbb{Z}[\sqrt{-5}]$ one has

$$9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5}).$$

The elements $3, 2 \pm \sqrt{-5}$ are irreducile in $\mathbb{Z}[\sqrt{-5}]$. We show more generally that every element of norm 9 in $\mathbb{Z}[\sqrt{-5}]$ is irreducile. Let $\alpha \in \mathbb{Z}[\sqrt{-5}]$, $N(\alpha) = 9$ and assume $\alpha = \beta\gamma$ with $\beta, \gamma \in \mathbb{Z}[\sqrt{-5}]$. Then

$$9 = N(\alpha) = N(\beta)N(\gamma),$$

so we have the following possibilities.

- Let $\beta = a + b\sqrt{-5}$, $a, b \in \mathbb{Z}$. If $N(\beta) = 1$ then $a^2 + 5b^2 = 1$, which implies $\beta = \pm 1$, so β is a unit in $\mathbb{Z}[\sqrt{-5}]$.
- If $N(\beta) = 3$ then $a^2 + 5b^2 = 3$ has no integer solutions.

- If $N(\beta) = 9$ then, as above, γ is a unit.

So either β or γ is a unit in $\mathbb{Z}[\sqrt{-5}]$. Also, 3 and $2 + \sqrt{-5}$ are not associated in $\mathbb{Z}[\sqrt{-5}]$ (same for 3 and $2 - \sqrt{-5}$); if there is a unit $u = a + b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ such that

$$3 = u(2 + \sqrt{-5}),$$

then one has

$$\begin{cases} 2a - 5b = 3 \\ 2b + a = 0 \end{cases}$$

which has no solutions in $\mathbb{Z}$.

b. $\mathbb{Z} \subseteq \mathbb{Z}[\sqrt{-5}]$ is an integral extension, so $\dim(\mathbb{Z}[\sqrt{-5}]) = \dim \mathbb{Z} = 1$.

4 Let $0 \to N \xrightarrow{f} K$ be exact. It induces

$$S^{-1}K \xrightarrow{S^{-1}f} S^{-1}K$$

sending $n/s \in S^{-1}N$ to $f(n)/s \in S^{-1}K$. We have that

$$\begin{aligned} f(n)/s = 0 &\iff \exists s' \in S \text{ such that } s'f(n) = 0 \\ &\iff f(s'n) = 0 \\ &\overset{f \text{ inj.}}{\iff} s'n = 0 \\ &\implies n/s = 0. \end{aligned}$$

So $S^{-1}f$ is injective.

5 a. We have $A/(X_1, \dots, X_n) \simeq K \neq 0$; for all $i = 1, \dots, n$

$$A/(X_1, \dots, X_i) \simeq K[X_{i+1}, \dots, X_n] =: B$$

and X_{i+1} is regular in B.

b. Observe the following fact: if $\underline{a} := a_1, \dots, a_n$ is M-regular and for some $m_i \in M$ one has $\sum_{i=1}^n a_i m_i = 0$, then $m_i \in (\underline{a})M$ for all i. In fact, $m_n \in (a_1, \dots, a_{n-1})M \subseteq (\underline{a})M$ since by definition a_n is $M/(a_1, \dots, a_{n-1})M$-regular. If I could permute the sequence I conclude, but the permuted sequence is NOT in general regular; but we can surely say that $a_1, \dots, a_{n-1}$ is M-regular. Write

$$m_n = \sum_{i=1}^{n-1} a_i u_i, \quad u_i \in M.$$

Then

$$\sum_{i=1}^{n-1} a_i(m_i + a_n u_i) = 0.$$

It follows that

$$m_{n-1} + a_n u_{n-1} \in (a_1, \dots, a_{n-2})M,$$

so $m_{n-1} \in (a_1, \dots, a_{n-2}, a_n)M \subseteq (\underline{a})M$. Conclude by induction on n.
To prove the claim in b., note that it suffices to show that $a_1^d, a_2, \dots, a_n$ is M-regular for all $d > 0$: on $M/a_1^d M$ $a_2, \dots, a_n$ is $M/a_1^d M$-regular. So we can proceed by induction after noting that since a_1 is M-regular, then a_1^d is M-regular, since $a_1^d m = 0$ implies $a_1^{d-1} m = 0$ and so on.
It remains to show that if $i > 1$, for all $\mu_j \in M$ so that

$$a_i \mu_i = a_1^d \mu_1 + \dots + a_{i-1}\mu_{i-1}$$

one has

$$\mu_i \in (a_1^d, a_2, \dots, a_{i-1})M.$$

By induction on d one has $\mu_i \in (a_1^{d-1}, \dots, a_{i-1})M$. Write

$$\mu_i = a_1^{d-1}\zeta_1 + a_2\zeta_2 + \dots + a_{i-1}\zeta_{i-1}, \quad \zeta_j \in M \tag{1}$$

Then

$$a_1^{d-1}(a_1\mu_1 - \zeta_1 a_i) + \sum_{j=2}^{i-1} a_j(\mu_j - \zeta_j a_i) = 0;$$

by the fact we proved above it follows that

$$a_1\mu_1 - a_i\zeta_1 \in (a_1^{d-1}, \dots, a_{i-1})M,$$

so

$$a_i\zeta_1 \in (a_1, \dots, a_{i-1})M.$$

Since $\underline{a}$ is M-regular,

$$\zeta_1 \in (a_1, \dots, a_{i-1})M$$

and one concludes by substitute ζ_1 in (1).

c. Let $A = K[X, Y, Z]$; the sequence X^3, XYZ is not A-regular, since $X^2 \cdot XYZ \in (X^3)$, but $X^2 \notin (X^3)$.