D-MATH
HS 2019
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Solutions 8

Commutative Algebra

1 Let $P = T^n + a_{n-1}T^{n-1} + \cdots + a_0$.

a. $[T]$, the class of T in B is integral over A, since

$$[T]^n + a_{n-1}[T]^{n-1} + \cdots + a_0 = 0$$

in B. Integral elements form a ring, so B is integral over A. Also, $P([T]) = 0$ in B.

b. The set $\{1, [T], \ldots, [T]^{n-1}\}$ is an A-basis of B: Let $a'_i \in A$ so that $a'_0 + \cdots + a'_{n-1}[T]^{n-1} = 0$ in B. Then $P | a'_0 + \cdots + a'_{n-1}T^{n-1}$, which is not possible since $\deg P = n$. They are generators, since by the above equation $[T]^m \in < 1, [T], \ldots, [T]^{n-1} >_A$ for all $m \geq n$.

c. $P - \prod_{i=1}^n (T - X_i) = (a_{n-1} - e_{n-1})T^{n-1} + (a_{n-2} + e_{n-2})T^{n-2} + \cdots + a_0 + (-1)^n e_0$, where e_i are the symmetric polynomials

$$e_0 = X_1 \ldots X_n$$
$$\vdots$$
$$e_{n-1} = X_1 + \cdots + X_n.$$

Then $I = (e_{n-1} - a_{n-1}, e_{n-2} + a_{n-2}, \ldots, e_0 + (-1)^n a_0)$. Note that we have the sequence of A-free extensions

$$\begin{array}{c} A[e_{n-1}, \ldots, e_0] = A[X_1, \ldots, X_n] \\ \vdots \\ A[e_{n-1}, \ldots, e_0][X_1][X_2] \\ | \\ A[e_{n-1}, \ldots, e_0][X_1] \\ | \\ A[e_{n-1}, \ldots, e_0] \end{array}$$

of rank over A $1, 2, \ldots, n$. So $A[X_1, \ldots, X_n]$ is a free $A[e_{n-1}, \ldots, e_0]$-module of rank $n!$. In particular, we have a A-linear morphism

$$A[X_1, \ldots, X_n] \simeq A[e_{n-1}, \ldots, e_0]^{n!}.$$

Therefore

$$\begin{aligned} C/I &\simeq (\oplus_{I=1}^{n!} A[e_{n-1},\dots,e_0])/(e_{n-1}-a_{n-1}, e_{n-2}+a_{n-2},\dots,e_0+(-1)^n a_0) \\ &\simeq \oplus_{I=1}^{n!}(A[e_{n-1},\dots,e_0]/(e_{n-1}-a_{n-1}, e_{n-2}+a_{n-2},\dots,e_0+(-1)^n a_0)) \\ &\simeq A^{n!}. \end{aligned}$$

d. If $Q := P - \prod_{i=1}^{n}(T - X_i)$, for all $i = 1,\dots,n$ one has $Q(X_i) = P(X_i)$ in $C/\mathfrak{m}$. So $[X_i]$ are the roots of P in $C/\mathfrak{m}$.

2 a. Let $x = c/b$, with $c, b \in A$, $b \neq 0$ and

$$(c/b)^m + a_{m-1}(c/b)^{m-1} + \dots + a_0 = 0,$$

with $a_i \in A$. For all $n \geq 0$, since $A[x]$ is A-finitely generated by $\{1,\dots,x^{m-1}\}$,

$$x^n \in A + Ax + \dots + Ax^{m-1}.$$

Hence $b^{m-1}x^n \in A$ for all n.

b. Consider the following ascendent chain of ideals of A:

$$(a) \subseteq (a, ax) \subseteq (a, ax, ax^2) \subseteq \dots$$

By noetherianity, there exists an $m \geq 1$ such that

$$(a, ax, \dots, ax^{m+1}) = (a, ax, \dots, ax^m).$$

In particular there exist $a_0,\dots,a_m \in A$ so that

$$ax^{m+1} = a_0 a + \dots + a_m a x^m$$

which implies

$$a(x^{m+1} - a_m x^m - \dots - a_0) = 0$$

in K. Since $a \neq 0$ and A is an integral domain

$$x^{m+1} = a_m x^m + \dots + a_0,$$

thus $A[x]$ is generated over A by $1,\dots,x^m$, so x is integral over A.

3 a. Let $a/b \in K$ integral over A with $n \geq 1$ and $a_i \in A$ such that

$$(a/b)^n + a_{n-1}(a/b)^{n-1} + \dots + a_0 = 0.$$

Multiplying by $1/t^n$ one has

$$(a/tb)^n + a_{n-1}/t(a/tb)^{n-1} + \dots + a_0/t^n = 0.$$

By assumption $a/tb \in A_t$, so we can write $a/tb = c_1/t^m$ for some $c_1 \in A$, $m \geq 0$. Then $a/b = c_1/t^{m-1}$ and

$$(c_1/t^{m-1})^n + a_{n-1}(c_1/t^{m-1})^{n-1} + \cdots + a_0 = 0.$$

By multiplying by $(t^{m-1})^n$ one gets that $c_1^n \in (t)$, which implies that $c_1 \in (t)$, since A/tA has no non-trivial nilpotents. We can then write $a/b = c_2/t^{m-2}$ for some $c_2 \in A$. By an inductive argument, we obtain $a/b = c_m/t^{m-m} = c_m \in A$.

b. Let $A = \mathbb{C}[X,Y,Z]/(XZ - Y(Y+1))$ and $t = X$. Then

$$\begin{aligned} A/tA &\simeq \frac{(\mathbb{C}[X,Y,Z]/(XZ - Y(Y+1)))}{(X, XZ - Y(Y+1))/(XZ - Y(Y+1))} \\ &\simeq \mathbb{C}[X,Y,Z]/(X, XZ - Y(Y+1)) \\ &\simeq \mathbb{C}[X,Y,Z]/(X, Y(Y+1)) \\ &\simeq \mathbb{C}[Y,Z]/(Y(Y+1)). \end{aligned}$$

Since in $\mathbb{C}[Y,Z]/(Y(Y+1))$, $Y = -Y^2$, it is generated over $\mathbb{C}[Z]$ by $1, Y$, so every element has a representative of the form

$$f_0(Z) + Yf_1(Z),$$

with $f_0, f_1 \in \mathbb{C}[Z]$. If for some $n \geq 0$ $(f_0(Z) + Yf_1(Z))^n = Y(Y+1)g(Y,Z)$, with $g(Y,Z) \in \mathbb{C}[Y,Z]$, then by evaluating at $Y = 0$ one has that $f_0(Z)^n = 0$, $f_0(Z) = 0$. Also $f_1(Z) = 0$ by evaluating at $Y = -1$. Hence A/tA has no non-trivial nilpotents. The localization A_t is isomorphic to

$$\mathbb{C}[X,Y,Z]_X/(XZ - Y(Y+1))_X$$

(since the localization is an exact functor). Also,

$$\begin{aligned} \mathbb{C}[X,Y,Z]_X/(XZ - Y(Y+1))_X &\simeq \mathbb{C}[X,Y,Z]_X/(Z - \frac{Y(Y+1)}{X})\mathbb{C}[X,Y,Z]_X \\ &\simeq \mathbb{C}[X, \frac{1}{X}, Y, Z]_X/(Z - \frac{Y(Y+1)}{X}) \\ &\simeq \mathbb{C}[X, \frac{1}{X}, Y]. \end{aligned}$$

Claim: $\mathbb{C}[X, \frac{1}{X}, Y]$ is integrally closed.
Let $h/l = h(X,Y)/l(X,Y) \in \mathbb{C}(X,Y)$ be integral over $\mathbb{C}[X, \frac{1}{X}, Y]$, with $(h,l) = 1$ ($\mathbb{C}[X,Y]$ is an UFD). Let $n \geq 1$ and $f_i \in \mathbb{C}[X, \frac{1}{X}, Y]$ such that

$$(h/l)^n + f_{n-1}(h/l)^{n-1} + \cdots + f_0 = 0.$$

Let $m := \min\{s : X^s f_i \in \mathbb{C}[X,Y]\ \forall i\}$. Multiply the above equation by $X^m l^n$ and let $l_i = X^m f_i l^{n-i} \in \mathbb{C}[X,Y]$ for all i. We obtain

$$X^m h^n + l_{n-1}h^{n-1} + \cdots + l_0 = 0,$$

so $X^m h^n \in (l)$. If $m = 0$, then $l|h$, so $h/l \in \mathbb{C}[X,Y]$. Otherwise, evaluate at $X = 0$ and get $l = Xl_1$, with $l_1 \in \mathbb{C}[X,Y]$. Therefore

$$(h/Xl_1)^n + f_{n-1}(h/Xl_1)^{n-1} + \cdots + f_0 = 0.$$

As above, we have that $l_1 = Xl_2$ for some $l_2 \in \mathbb{C}[X,Y]$, or $l_1|h$ so $h/Xl_1 \in \mathbb{C}[X, \frac{1}{X}, Y]$ if $m = 1$, and so on until $h/l = h/X^m l_m$ with $l_m|h$, hence $h/l \in A_t$.

4 a. Let $C = B[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/I$ where I is generated by the coefficients of $P - \prod_{i=1}^m (T - X_i)$ and $Q - \prod_{i=1}^n (T - Y_i)$. Then $a_i = X_i \bmod I$ and $b_i = Y_i \bmod I$. By exercise 1, C is integral over B and PQ splits in C as

$$PQ = \prod_{i=1}^m (T - a_i) \prod_{i=1}^n (T - b_i).$$

Let A' be the integral closure of C in A, so $a_i, b_j \in A'$. Also, A' is a subring of C, hence the coefficients of P and of Q are in $A' \cap B = A$, since A is integrally closed in B.

b. Let $P \in B[T]$ integral over $A[T]$, with $s \geq 1$ and $a_i \in A[T]$ so that

$$P^s + a_{s-1}(T)P^{s-1} + \cdots + a_0(T) = 0.$$

Let $Q = T^m + P$ with $m > \max(n, \deg(a_{s-1}), \ldots, \deg(a_0))$. Let $p(x) := x^s + a_{s-1}(T)x^{s-1} + \cdots + a_0(T)$, so $p(P) = 0$; then Q is a root of $p(x - T^m)$, and

$$p(x - T^m) = x^s + b_{s-1}(T)x^{s-1} + \cdots + b_0(T),$$

where $b_0(T) = p(-T^m) \in A[T]$ is monic up to a sign, by the choice of m. Therefore

$$Q(Q^{s-1} + b_{s-2}(T)Q^{s-2} + \cdots + b_1(T)) = -b_0(T) \in A[T].$$

Then $Q^{s-1} + b_{s-2}(T)Q^{s-2} + \cdots + b_1(T)$ is monic and by part a. $Q = P + T^m \in A[T]$, then $P \in A[T]$.

5 Let $s = \operatorname{rank} M \leq m$ and we can assume $\underline{v_1}, \ldots, \underline{v_s}$ are linearly independent:

$$< \underline{v_1}, \ldots, \underline{v_s} > = < \underline{v_1}, \ldots, \underline{v_m} > \subseteq \mathbb{Q}^r.$$

Claim: $Y^{\underline{v_1}}, \dots, Y^{\underline{v_s}}$ are algebraically independent over K.
Let $f = \sum_{u \in \mathbb{N}^s} a^u X^u \in K[X_1, \dots, X_s]$, then

$$f(Y^{\underline{v_1}}, \dots, Y^{\underline{v_s}}) = \sum a^u Y^{\underline{u_1 v_1}}, \dots, Y^{\underline{u_s v_s}}.$$

In the last sum, if two monomials are equal, since $\underline{v_1}, \dots, \underline{v_s}$ are linearly independent, one has that the u_i are equal. This implies that if $f(Y^{\underline{v_1}}, \dots, Y^{\underline{v_s}}) = 0$ then $f = 0$.
Now, $K \subseteq K(Y^{\underline{v_1}}, \dots, Y^{\underline{v_s}})$ is a purely algebraic extension (i.e. $K(Y^{\underline{v_1}}, \dots, Y^{\underline{v_s}})$ is the smallest field which contains K and the algebraically independent elements $Y^{\underline{v_1}}, \dots, Y^{\underline{v_s}}$).
Claim: The extension $A \subseteq \mathrm{frac}(A)$, $A = K[Y^{\underline{v_1}}, \dots, Y^{\underline{v_s}}]$ is algebraic.
Let $j > s$ and write $v_j = \sum_{i=1}^{s} \lambda_i v_i$ with $\lambda_i \in \mathbb{Q}$. Write $\lambda_i = a_i/b_i$ and let $b = \prod_{i=1}^{s} b_i$. Then $bv_j = \sum_{i=1}^{s} c_i v_i$ with $c_i \in \mathbb{Z}$ and

$$(Y^{\underline{v_j}})^b = \prod_{i=1}^{s} (Y^{\underline{v_i}})^{c_i}.$$

This proves that the trascendence degree of $\mathrm{frac}(B)/K$ is s.

6 The minimal primes $\wp$ of A have height 0, so $\dim(A/\wp) \le \dim A$ and

$$\dim A \ge \max\{\dim(A/\wp) : \wp \text{ minimal prime of } A\}.$$

On the other hand, for a prime $\wp_1$ of A

$$\dim(A/\wp_1) = \max\{t : \exists \wp \subseteq A \ \exists \text{ chain of lenght } t \ \wp_1 \subset \cdots \subset \wp\}.$$

Let $\wp_0$ be a minimal prime of A such that $\dim(A/\wp_0)$ is the maximum, say t:

$$\wp_0 \subset \cdots \subset \wp_t = \wp.$$

Further, let $\wp' \in \mathrm{Spec} A$ and $\wp'_0 \subset \cdots \subset \wp'$ a chain of maximal lenght; in particulat $\wp'_0$ is a minimal prime. We've shown that

$$ht(\wp') \le \dim(A/\wp'_0) \le \dim(A/\wp_0)$$

for all $\wp' \in \mathrm{Spec} A$.