D-MATH
HS 2019
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Solutions 9

Commutative Algebra

1 a. Since $\ker f$ is a submodule of M, $\ell(\ker f) \leq \ell(M) < \infty$. Moreover, by $\operatorname{im} f \simeq M/\ker f$ one has $\ell(\operatorname{im} f) = \ell(M) - \ell(\ker f) < \infty$.

b. Assume that f is injective. Since $\ell(M) < \infty$, in particular M is artinian. Then the following descendent chain stabilizes:

$$\operatorname{im} f \supseteq \operatorname{im} f^2 \supseteq \dots,$$

that is, there exists $t \geq 1$ such that $\operatorname{im} f^t = \operatorname{im} f^{t+1}$. Let $n \in M$ and let $m \in M$ such that $f^t(n) = f^{t+1}(m)$. Then

$$\begin{aligned} & f^t(n) - f^t(f(m)) = 0 \\ \Longrightarrow & f^t(n - f(m)) = 0 \\ \overset{f \text{ injective}}{\Longrightarrow} & n = f(m) \\ \Longrightarrow & n \in \operatorname{im} f. \end{aligned}$$

If f is surjective, consider the analogous ascendent chain

$$\ker f \subseteq \ker f^2 \subseteq \dots$$

and use the fact that M is noetherian.

c. Let n as above so that $\operatorname{im} f^n = \operatorname{im} f^m$ for all $m \geq n$. Let $m \in M$ and $m' \in M$ such that $f^{n+n}(m') = f^n(m)$. Then

$$m = (m - f^n(m')) + f^n(m')$$

and

$$f^n(m - f^n(m')) = f^n(m) - f^{n+n}(m') = 0,$$

so $m - f^n(m') \in \ker f^n$. Hence

$$M = \ker f^n + \operatorname{im} f^n.$$

2 Consider the chains of exercise 1 and pick p and q minimal such that they stabilize (M is both noetherian and artinian since of finite length).

Assume $p \geq q$, so $\operatorname{im} u^p = \operatorname{im} u^q$ and $\ker u^p \subseteq \ker u^q$. By exercise 1.a

$$\begin{aligned}\ell(M) &= \ell(\ker u^p) + \ell(\operatorname{im} u^p)\\ &= \ell(\ker u^q) + \ell(\operatorname{im} u^q),\end{aligned}$$

which implies

$$\ell(\ker u^p) = \ell(\ker u^q)$$

and so by the properties of the length, $\ker u^p = \ker u^q$. But p is minimal, so $p \leq q$, then $p = q$.

d. By exercise 1.c we have $M = \ker u^p + \operatorname{im} u^p$. Let $x \in \ker u^p \cap \operatorname{im} u^p$ and let y such that $x = u^p(y)$ and so $u^p(x) = u^{2p}(y) = 0$. Then $y \in \ker u^{2p} = \ker u^p$, so $x = u^p(y) = 0$.

3 Let $n > 1$; then since $\mathbb{Q}$ is divisible $n\mathbb{Q} = \mathbb{Q}$, but there is no element a in $n\mathbb{Z}$ such that $(1+a)\mathbb{Q} = 0$.

4 Let $I = (a_1, \ldots, a_n)$. Since $I = II$, by Nakayama's Lemma there is an element $a \in I$ such that $(1+a)I = 0$. In particular for every $i = 1, \ldots, n$, $(1+a)a_i = 0$, so $a_i = -aa_i \in (a)$, which implies

$$I = (a).$$

Now, since $I = I^2$, there is a $u \in A^\times$ such that $a = ua^2$. Choose $e := ua$, then $I = (e)$ and $e^2 = u^2a^2 = u^2u^{-1}a = ua = e$.

5 a. Let $\Phi : M \longrightarrow \prod_{\mathfrak{m} \subseteq A \text{ maximal}} M_\mathfrak{m}$. First of all, note that if M is a simple module (i.e. $\ell(M) = 1$), say $M \simeq A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A, then $M_\mathfrak{m} \simeq A/\mathfrak{m}$, since $A/\mathfrak{m}$ is a field. Moreover, if $\mathfrak{m}' \neq \mathfrak{m}$, then

$$M_{\mathfrak{m}'} \simeq (A/\mathfrak{m})_{\mathfrak{m}'} \simeq A_{\mathfrak{m}'}/\mathfrak{m}_{\mathfrak{m}'} = 0,$$

since $\mathfrak{m} \not\subseteq \mathfrak{m}'$. In particular, if $\mathfrak{m}'$ and $\mathfrak{m}''$ are distinct maximal ideals, then $(M_{\mathfrak{m}'})_{\mathfrak{m}''} = 0$.
Now, let $n := \ell(M)$ and pick a decomposition series

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0.$$

By localizing at $\mathfrak{m}$ we get

$$M_\mathfrak{m} = (M_0)_\mathfrak{m} \supset (M_1)_\mathfrak{m} \supset \cdots \supset (M_n)_\mathfrak{m} = 0.$$

The quotients M_i/M_{i+1} are simple, so by the above remarks

$$(M_i/M_{i+1})_\mathfrak{m} = \begin{cases} M_i/M_{i+1} & \text{if } \mathfrak{m} = (0 :_A M_i/M_{i+1})\\ 0 & \text{otherwise.}\end{cases}$$

From this, we see that $M_{\mathfrak{m}}$ has a decomposition series corresponding to the subseries of the one for M, obtained by keeping only those $(M_i)_{\mathfrak{m}}$ such that $M_i/M_{i+1} \simeq A/\mathfrak{m}$. Also, if $\mathfrak{m}' \neq \mathfrak{m}''$, then $(M_{\mathfrak{m}'})_{\mathfrak{m}''} = 0$.
Consider now a maximal ideal $\mathfrak{m}'$ and the localization of Φ:

$$\Phi_{\mathfrak{m}'} : M_{\mathfrak{m}'} \longrightarrow \prod_{\mathfrak{m} \subseteq A \text{ maximal}} (M_{\mathfrak{m}})_{\mathfrak{m}'} = (M_{\mathfrak{m}'})_{\mathfrak{m}'} = M_{\mathfrak{m}'}.$$

Then $\Phi_{\mathfrak{m}'} = \mathrm{id}_{M_{\mathfrak{m}'}}$ for every maximal ideal $\mathfrak{m}'$. In particular $\Phi_{\mathfrak{m}'}$ is an isomorphism of $A_{\mathfrak{m}'}$-modules for every maximal ideal $\mathfrak{m}'$; but the localization is a flat module, so the above implies that Φ is an isomorphism of A-modules (slogan: "being an isomorphism is local property").

b. Since A is artinian, it has finite length, and there are only finitely many maximal ideals, so by part a. we get an isomorphism of A-modules

$$\Phi : A \xrightarrow{\simeq} \prod_{\mathfrak{m} \subseteq A \text{ maximal}} A_{\mathfrak{m}} \overset{A\text{-mod.}}{\simeq} \bigoplus_{\mathfrak{m} \subseteq A \text{ maximal}} A_{\mathfrak{m}}.$$

Since each map $A \to A_{\mathfrak{m}}$ is a morphism of rings, the isomorphism Φ is actually an isomorphism of rings.