COUNTING PRIMITIVE INTEGER VECTORS USING ESKIN-MCMULLEN

MANUEL W. LUETHI

ABSTRACT. Notes for preparation of the toy case for Eskin McMullen

In what follows, $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $K = \mathrm{SO}_2(\mathbb{R})$,

$$A = \left\{ a_t = \begin{pmatrix} e^{-\frac{t}{2}} & \\ & e^{\frac{t}{2}} \end{pmatrix}; t \in \mathbb{R} \right\},$$

$$U = \left\{ u_t = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}; t \in \mathbb{R} \right\}.$$

For the choice of Haar measures, we equip $U \cong \mathbb{R}$ with the Lebesgue measure and $\mathrm{SO}_2(\mathbb{R}) \cong S^1$ with the spherical measure that assigns mesure 2π. Explicitly, this measure is given by the formula

$$\int_{\mathrm{SO}_2(\mathbb{R})} f(k) \mathrm{d}m_{\mathrm{SO}_2(\mathbb{R})}(k) = \int_0^{2\pi} f(k_\theta) \mathrm{d}\theta$$

for all $f \in C\big(\mathrm{SO}_2(\mathbb{R})\big)$, where $k_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$.

Lemma 1. $\mathbb{R}^2 \setminus \{0\} \cong U \backslash G$.

Proof. $G \curvearrowright \mathbb{R}^2 \setminus \{0\}$ jointly continuously via $g \cdot v = vg^{-1}$, where we identify $\mathbb{R}^2$ with the space of 1×2-matrices with entries in $\mathbb{R}$ (i.e. row-vectors). The action is transitive and $\mathrm{Stab}_G(e_2) = U$. □

As a corollary we obtain the following *disintegration* formula for the Haar measure on $\mathrm{SL}_2(\mathbb{R})$.

Corollary 1. *The map on* $C_c(\mathrm{SL}_2(\mathbb{R}))$ *given by*

$$f \mapsto \int_{\mathbb{R}^2} \int_U f(ug) \mathrm{d}u \mathrm{d}e_2 g$$

is a right-invariant functional on $\mathrm{SL}_2(\mathbb{R})$ *and thus induces a right-invariant Haar measure on* $\mathrm{SL}_2(\mathbb{R})$.

Proof. This was shown (exchanging left for right actions) earlier. □

Remark 1. *Every right Haar measure on* $G = \mathrm{SL}_2(\mathbb{R})$ *is a left Haar measure. This is shown as follows:*

(1) *Let m be any right Haar measure on G and let $g \in G$ fixed. Define a measure m_g on G by $m_g(B) = m(gB)$. As left multiplication by g is a homeomorphism of G, m_g is again a Borel measure which is finite on compacta and does not vanish on non-empty open sets. Moreover right-invariance of m yields $m_g(Bh) = m(gBh) = m(gB) = m_g(B)$ for all $h \in G$ and for all Borel sets B. Thus m_g is again a right Haar measure.*

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(2) *Uniqueness of the Haar measure up to positive scalar multiples implies that there exists* $\chi : G \to (0,\infty)$ *such that for all* $g \in G$ *holds* $m_g = \chi(g)m$. *One checks that* $\chi(gh) = \chi(g)\chi(h)$ *for all* $g, h \in G$. *Indeed for any non-empty open Borel set* B *in* G *we find* $\chi(gh)m(B) = m_{gh}(B) = m_g(hB) = \chi(g)m(hB) = \chi(g)m_h(B) = \chi(g)\chi(h)m(B)$ *and* $m(B) \neq 0$ *yields* $\chi(gh) = \chi(g)\chi(h)$.

(3) *As* $(0,\infty)$ *is an abelian group, the commutator subgroup of* G *is contained in the kernel of* χ. *But* G *is almost simple (i.e. has no infinite normal subgroups) and as* $(0,\infty)$ *has non non-trivial finite subgroups, it follows that* $\chi(g) = 1$ *for all* $g \in G$.

Instead of proving almost simplicity, we will show that the commutator subgroup $[G,G] \leq G$, *i.e. the group generated by all elements of the form* $ghg^{-1}h^{-1}$ *actually agrees with the full group* G. *To this end it suffices to show that* $[G,G]$ *contains all upper and lower unipotents. One calculates*

$$u_s a_t u_{-s} a_{-t} = u_s u_{-e^{-t}s} = u_{(1-e^{-t})s} \in [G,G]$$

and similarly for lower unipotents.

(4) *As* $\chi \equiv 1$, *it follows that* $m_g = m$ *for all* $g \in G$ *and thus* m *is in fact left-invariant. This proves the claim.*

Lemma 2 (Iwasawa decomposition)**.** *Let* $g \in \mathrm{SL}_2(\mathbb{R})$, *then there are unique* $k \in K$, $a \in A$, $u \in U$ *such that* $g = uak$. *We write* $k = k(g)$, $a = a(g)$ *and* $u = u(g)$. *We also denote by* $t_g \in \mathbb{R}$ *the number defined by* $a(g) = a_{t_g}$.

Proof. Consider $v = e_2 g$. Then there exists a unique $k \in \mathrm{SO}_2$ such that $vk = \|v\| e_2$. Choose $t \in \mathbb{R}$ such that $vka_t = e_2$. Then $e_2 k a_t = e_2$ implies $gka_t = u$ for some $u \in U$ and thus $g = ua_{-t}k$. Uniqueness follows from uniqueness of k, a_t and right-cancellation in groups. □

Lemma 3 (KAK-decomposition)**.** *Let* $g \in \mathrm{SL}_2(\mathbb{R})$. *Then there is a unique* t *and there are* $k, l \in \mathrm{SO}_2(\mathbb{R})$ *such that* $g = ka_{\pm t}l$. *In fact, for every* $\varepsilon > 0$ *the set*

$$V_\varepsilon = K\,\{a_t; |t| < \varepsilon\}\,K \quad (\varepsilon > 0)$$

is an open neighbourhood of the identity.

Proof. Let $S \subseteq \mathbb{R}^2 \setminus \{0\}$ be a circle, then Sg is an ellipse. Let $k \in \mathrm{SO}_2(\mathbb{R})$ sending the major axis of the ellipse to the real line. Now choose $t \in \mathbb{R}$ such that $Sgka_t$ is again a circle. As gka_t sends one circle to another circle, it follows from linearity that gka_t sends circles to circles (i.e. is a linear transformation that preserves lengths and thus in fact orthogonal). As $\det(gka_t) = 1$, it follows that $gka_t \in \mathrm{SO}_2(\mathbb{R})$.

It remains to show the uniqueness. Note that for $g = kal$ holds $S^1 g = S^1 al$ and the latter is in fact independent of the choice of the decomposition of g. The element a determines the ratio between the semi-minor axes, and thus is uniquely defined by the image of S^1 under g.

In order show that these sets form an open neighbourhood, look at the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ and consider the set $V_\varepsilon \cdot \mathbf{i}$. This is given by the (hyperbolic) ball determined by the piece $(e^{-\varepsilon}, e^{\varepsilon}) \cdot \mathbf{i}$ around $\mathbf{i}$ – because K moves points in $\mathbb{H}$ along the hyperbolic circle centered at $\mathbf{i}$ – and thus an open neighbourhood of $\mathbf{i}$. As the map $G \to \mathbb{H}$, $g \mapsto g \cdot \mathbf{i}$ is a quotient map, the preimage given by V_ε is an open subset of G. □

Proposition 1. *The set of primitive integer vectors in* $\mathbb{R}^2$ *is given by* $\Gamma \cdot e_2 = e_2\Gamma \cong \Gamma/\Gamma_\infty$, *where* $\Gamma_\infty = \Gamma \cap U$.

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, then $ad - bc = 1$ implies that c and d are coprime. Thus $e_2 g = \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$ is a primitive vector.

Let $v = (m, n) \in \mathbb{Z}^2$ be a primitive vector. Then m, n are coprime and thus there are $a, b \in \mathbb{Z}$ so that $am + bn = 1$. In particular $g = \left(\begin{smallmatrix} b & -a \\ m & n \end{smallmatrix}\right) \in \Gamma$ satisfies $v = e_2 g$. It follows that the primitive vectors are contained in $\Gamma \cdot e_2$ and using the definition of a group action, it follows that Γ acts transitively on the set of primitive vectors.

Finally $\mathrm{Stab}_\Gamma(e_2) = \mathrm{Stab}_G(e_2) \cap \Gamma = U \cap \Gamma$ implies the claim. $\square$

In order to prove the equidistribution of long horocycle orbits after twisting with elements in $\mathrm{SO}_2(\mathbb{R})$, we include the following elementary lemma.

Lemma 4. *Let $\varphi : (-\infty, 0) \to \mathbb{C}$ be a function and $a \in \mathbb{C}$. The following are equivalent:*

1. *For all $\varepsilon > 0$ there is some $T_\varepsilon \in (-\infty, 0)$ such that for all $t \leq T_\varepsilon$ holds $|\varphi(t) - a| < \varepsilon$ (i.e. $\varphi(t) \to a$ as $t \to -\infty$).*
2. *For every sequence $(t_n)_{n\in\mathbb{N}}$ of negative numbers satisfying $t_n \to -\infty$ holds $\varphi(t_n) \to a$ as $n \to \infty$.*

Proof. It is clear that the first item implies the second. In order to prove the opposite implication, assume the first item is false, i.e. there is some $\varepsilon > 0$ such that for all $T < 0$ there exists some $t \leq T$ with $|\varphi(t) - a| \geq \varepsilon$. Choose $t_1 < -1$ such that $|\varphi(t) - a| \geq \varepsilon$. Assume that we have found $0 > t_1 > \ldots > t_n$ such that $t_k - t_{k-1} < -1$ for all $1 < k \leq n$ and $|\varphi(t_k) - a| \geq \varepsilon$ for all $1 \leq k \leq n$. By assumption, there is some $t_{n+1} < t_n - 1$ such that $|\varphi(t_{n+1}) - a| \geq \varepsilon$ and proceed inductively. The sequence $(t_n)_{n\in\mathbb{N}}$ satisfies $t_n \to -\infty$ and $\varphi(t_n) \not\to a$. $\square$

Lemma 5. *For $t \in \mathbb{R}$ let $k_t \in \mathrm{SO}_2(\mathbb{R})$ arbitrary. The sets $\Gamma U a_{-t} k_t$ equidistribute in $\Gamma\backslash G$ as $t \to -\infty$ in the following sense. For all $f \in C_c(\Gamma\backslash G)$ holds*

$$\int_0^1 f(\Gamma u_s a_{-t} k_t) \mathrm{d}s \overset{t\to-\infty}{\longrightarrow} \int_{\Gamma/G} f(x) \mathrm{d}x.$$

Proof. The motivation behind this is that $\Gamma U a_{-t} = \Gamma a_{-t} U$ and thus Γa_{-t} has a periodic U-orbit of volume e^{-t}, because $\Gamma a_{-t} u_s = \Gamma a_{-t} \iff a_{-t} u_s a_t = u_{e^t s} \in \Gamma$. The proof that the sets $\Gamma U a_{-t}$ equidistribute is a result attributed to Sarnak, but can and in this course will be obtained by more elementary means via Margulis' "banana" trick, which is the content of the talk preceding this one. It remains to show that this result is not affected by translation of the orbit using elements in $\mathrm{SO}_2(R)$.

Using the preceding lemma, it suffices to show that for every sequence $(t_n)_{n\in\mathbb{N}}$ and for every $f \in C_c(\Gamma\backslash G)$ we have

$$\int_0^1 f(\Gamma u_s a_{-t_n} k_{t_n}) \mathrm{d}s \to \int_{\Gamma/G} f(x) \mathrm{d}x.$$

As $\mathrm{SO}_2(\mathbb{R})$ is compact, every sequence contains a subsequence such that – keeping the same indices – $k_{t_n} \to k \in \mathrm{SO}_2(\mathbb{R})$. We will write k_n and a_n instead of k_{t_n} and a_{-t_n}. Let now $\varepsilon > 0$ arbitrary. Using the triangle inequality, we have

$$\begin{aligned} \left| \int_0^1 f(\Gamma u_s a_n k_n) - \int_{\Gamma/G} f(x) \mathrm{d}x \right| &\leq \left| \int_0^1 f(\Gamma u_s a_n k_n) - \int_0^1 f(\Gamma u_s a_n k) \right| \\ &+ \left| \int_0^1 k \cdot f(\Gamma u_s a_n) - \int_{\Gamma/G} k^{-1} \cdot f(x) \mathrm{d}x \right|. \end{aligned}$$

As f is uniformly continuous, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ holds $|f(xk_n) - f(xk)| < \frac{\varepsilon}{2}$ for all $x \in G/\Gamma$. Note that $k^{-1} \cdot f \in C_c(X)$, so that as $\Gamma U a_n$ equidistributes in $\Gamma\backslash G$, we find that there is some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$

we have
$$\left|\int_0^1 k^{-1}\cdot f(\Gamma u_s a_n) - \int_{\Gamma/G} k^{-1}\cdot f(x)\mathrm{d}x\right| < \frac{\varepsilon}{2}.$$
Setting $N = \max\{N_1, N_2\}$ shows that
$$n \geq N \implies \left|\int_0^1 f(\Gamma u_s a_n k_n) - \int_{\Gamma/G} f(x)\mathrm{d}x\right| < \varepsilon.$$
Now we invoke the argument from the class. Every subsequence contains a subsequence which converges to $\int_{\Gamma/G} f(x)\mathrm{d}x$, hence the whole sequence converges to this limit and thus equidistribution follows. □

Corollary 2. *Let g_n be a sequence in $\mathrm{SL}_2(\mathbb{R})$ such that $a(g_n) = a_{-t_n}$ for $t_n \to -\infty$ (in particular $Ug_n \to \infty$ in $U\backslash G$). Then $\Gamma U g_n$ equidistributes in $\Gamma\backslash G$.*

Proof. $\Gamma U g_n = \Gamma U a_{-t_n} k(g_n)$, hence it follows from the preceding statement. □

Proposition 2. *In fact, the rate of equidistribution (for fixed f) is independent of k_t, i.e. for all $\varepsilon > 0$ there is a $T_\varepsilon < 0$ with the right property for all $k_t \in \mathrm{SO}_2(\mathbb{R})$.*

Proof. This is the same argument as before. If this was not true, there were an $\varepsilon > 0$ such that for all $T < 0$ one could find some $t \leq T$ and some $k_t \in \mathrm{SO}_2(\mathbb{R})$ such that
$$\left|\int_0^1 f(\Gamma u_s a_{-t} k_t)\mathrm{d}t - \int_{G/\Gamma} f(x)\mathrm{d}x\right| \geq \varepsilon.$$
This contradicts the preceding discussion. □

For the calculations to follow, we will often employ the folding/unfolding trick. Let $H \leq G$ be a closed subgroup such that $\Gamma_\infty \leq H$ and assume that $\Gamma_\infty H \subseteq \Gamma_\infty\backslash G$ is a closed subset. Assume furthermore that $\Gamma_\infty\backslash H$ admits an H-invariant measure $m_{\Gamma_\infty/H}$ and that $H\backslash G$ admits a G-invariant measure $m_{H/G}$. Then the map sending $f \in C_c(\Gamma_\infty\backslash G)$ to
$$\int_{H/G}\int_{\Gamma_\infty/H} f(\Gamma_\infty h g)\mathrm{d}m_{\Gamma_\infty/H}(\Gamma_\infty h)\mathrm{d}m_{H/G}(Hg)$$
is a well-defined G-invariant measure on $\Gamma_\infty\backslash G$.

Indeed the main point is that the map which send $f \in C_c(\Gamma_\infty\backslash G)$ to $Tf : H\backslash G \to \mathbb{C}$ defined by
$$Tf(Hg) = \int_{\Gamma_\infty/H} f(\Gamma_\infty h g)\mathrm{d}m_{\Gamma_\infty/H}(\Gamma_\infty h)$$
is well-defined and satisfies $Tf \in C_c(H\backslash G)$. The fact that Tf is well-defined as a function on $H\backslash G$ follows immediately from H-invariance of $m_{\Gamma_\infty/H}$. Let $g \in G$ such that $Tf(Hg) \neq 0$, then there is some $h \in H$ such that $f(\Gamma_\infty h g) \neq 0$ and thus $\Gamma_\infty h g \in \mathrm{supp}\,(f)$, so that $Hg \subseteq p_H(\mathrm{supp}\, f)$, where $p_H : \Gamma_\infty\backslash G \to H\backslash G$ denotes the canonical projection $\Gamma_\infty g \mapsto Hg$. As the latter is continuous, it follows that $p_H(\mathrm{supp}\, f)$ is compact and thus $\mathrm{supp}\, Tf$ is compact. It remains to show that Tf is continuous. Indeed Let $g, g_n \in G$ such that $Hg_n \to Hg$, i.e. there is a sequence of $h_n \in H$ such that $h_n g_n \to g$. Continuity of the projection implies that $\Gamma_\infty h_n g_n \to \Gamma_\infty g$. As argued above, we have $Tf(Hg_n) = Tf(Hh_n g_n)$. Define $f_n(\Gamma_\infty h) = f(\Gamma_\infty h h_n g_n)$, $f^*(\Gamma_\infty h) = f(\Gamma_\infty h g)$ (for $\Gamma_\infty h \in \Gamma_\infty\backslash H$). Then continuity of f implies that $f_n \to f^*$ pointwise as $n \to \infty$. Moreover, continuity of the action $G \curvearrowright \Gamma_\infty\backslash G$ implies that there exists a compact neighbourhood $V \subseteq G$ of the identity and some $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ holds $\mathrm{supp}\, f_n \subseteq (\mathrm{supp}\, f^*)V$. Again by continuity of the action, the subset $(\mathrm{supp}\, f^*)V$ is a compact subset of $\Gamma_\infty\backslash G$ and thus the function $\psi = \|f\|_\infty \mathbb{1}_{(\mathrm{supp}\, f^*)V \cap \Gamma_\infty H}$ is an integrable function on $\Gamma_\infty\backslash H$

satisfying $|f_n| \leq \psi$ pointwise. Hence dominated convergence yields $Tf(Hg_n) \to Tf(Hg)$ as $n \to \infty$.

One can show that up to scalar multiples there is at most one G-invariant Borel measure on $\Gamma_\infty \backslash G$.

Corollary 3. *Let m_G be the Haar measure on G induced by the disintegration via the action of G on $\mathbb{R}^2 \setminus \{0\}$ (cf. Corollary 1). Let $m_{\Gamma/G}$ be the G invariant measure on $\Gamma\backslash G$ induced by m_G. Let $m_{\Gamma_\infty/U}$ be the U-invariant measure on $\Gamma_\infty\backslash U$ induced by the disintegration via the action of U on ΓU. Then*

$$\int_{U/G}\int_{\Gamma_\infty/U} f(\Gamma_\infty ug)\mathrm{d}m_{\Gamma_\infty/U}(\Gamma_\infty u)\mathrm{d}m_{U/G}(Ug) = \int_{\Gamma/G}\sum_{\Gamma_\infty\gamma\in\Gamma_\infty/\Gamma} f(\Gamma_\infty\gamma g)\mathrm{d}m_{\Gamma/G}(\Gamma g).$$

Proof. Using uniqueness (up to scalar multiples) of the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\Gamma_\infty\backslash G$, it suffices to find a subset of $\Gamma_\infty\backslash G$ which is given the same non-zero measure by the two formulae. To this end (similarly to what happens in the end), let $E \subseteq \Gamma_\infty\backslash G$ be the set $E = \left\{\Gamma_\infty u_s a_t k; |s| < \frac{1}{2}, t < 0\right\}$. Write $E' = \left\{u_s a_t k; |s| < \frac{1}{2}, t < 0\right\}$. The sets E and E' are injective for the quotient maps $\Gamma_\infty g \mapsto \Gamma g$ and $g \mapsto \Gamma g$ respectively. Hence we obtain

$$\begin{aligned}
\int_{\Gamma/G}\sum_{\Gamma_\infty\gamma\in\Gamma_\infty/\Gamma} \mathbb{1}_E(\Gamma_\infty\gamma g)\mathrm{d}m_{\Gamma/G}(\Gamma g) &= \int_{\Gamma/G} \mathbb{1}_{\Gamma E'}(\Gamma g)\mathrm{d}m_{\Gamma/G}(\Gamma g)\\
&= \int_{U/G}\int_U \mathbb{1}_{E'}(ug)\mathrm{d}m_U(u)\mathrm{d}m_{U/G}(Ug)\\
&= \int_{U/G}\int_{\Gamma_\infty/U} \mathbb{1}_E(ug)\mathrm{d}m_{\Gamma_\infty/U}(\Gamma_\infty u)\mathrm{d}m_{U/G}(Ug).
\end{aligned}$$

□

In the case at hand, we are given the two quotients as in the diagram

$$\begin{tikzcd}
& \Gamma_\infty\backslash G \arrow[dl] \arrow[dr] & \\
U\backslash G & & \Gamma\backslash G
\end{tikzcd}$$

instead of just one and our goal is to use the disintegration with respect to both of these quotients. The count of interest is given by the number of points along the Γ-orbit of U in the quotient space on the left, whereas the U-orbit of Γ is known to equidistribute (when expanded by the geodesic flow).

Proposition 3 (Average counting result)**.** *Let $r > 0$ and define $F_r : \Gamma\backslash G \to \mathbb{R}$ by*

$$F_r(\Gamma g) = \frac{1}{\mathrm{vol}\big(B_r(0)\big)}|e_2\Gamma g \cap B_r(0)|.$$

Then

$$F_r \mathrm{d}m_{\Gamma/G} \overset{r\to\infty\star}{\longrightarrow} \frac{\mathrm{vol}(\Gamma_\infty\backslash U)}{\mathrm{vol}(\Gamma\backslash G)}\mathrm{d}m_{\Gamma/G}$$

in the weak-star topology. Here $m_{\Gamma/G}$ is a finite (not necessarily probability) G-invariant measure on $\Gamma\backslash G$ induced by $m_{U/G}$ and m_U.

Proof. Let $f \in C_c(\Gamma\backslash G)$, then

$$\begin{aligned}
\int_{\Gamma/G} f(x)F_r(x)\mathrm{d}m_{\Gamma/G}(x) &= \frac{1}{\mathrm{vol}\big(B_r(0)\big)}\int_{\Gamma/G} f(\Gamma g)\sum_{\Gamma_\infty\gamma\in\Gamma_\infty/\Gamma} \mathbb{1}_{B_r(0)}(e_2\gamma g)\mathrm{d}\Gamma g\\
&= \frac{1}{\mathrm{vol}\big(B_r(0)\big)}\int_{\Gamma_\infty/G} f(\Gamma g)\mathbb{1}_{B_r(0)}(e_2 g)\mathrm{d}\Gamma_\infty g
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{U/G} \int_{\Gamma_\infty/U} f(\Gamma u g) \mathbb{1}_{B_r(0)}(e_2 g) \mathrm{d}\Gamma_\infty u \mathrm{d}Ug \\
&= \frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{U/G} \mathbb{1}_{B_r(0)}(e_2 g) \int_{\Gamma_\infty/U} f(\Gamma u g) \mathrm{d}\Gamma_\infty u \mathrm{d}Ug \\
&= \frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{U/G} \mathbb{1}_{B_r(0)}(e_2 g) \int_0^1 f(\Gamma u_s g) \mathrm{d}s \mathrm{d}Ug.
\end{aligned}
$$

Let $g \in G$, then $\|e_2 g\| = \|e_2 a(g)\| = e^{\frac{t_g}{2}}$. Let $\varepsilon > 0$ and choose $T_\varepsilon > 0$ so that

$$
\left| \int_0^1 f(\Gamma u_s g) \mathrm{d}s - \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int_{\Gamma/G} f(x) \mathrm{d}x \right| < \varepsilon
$$

whenever $t_g > T_\varepsilon$ and let $r_\varepsilon = e^{\frac{T_\varepsilon}{2}}$. Then for ever $r > r_\varepsilon$ we obtain

$$
\begin{aligned}
&\frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{U/G} \mathbb{1}_{B_r(0)}(e_2 g) \int_0^1 f(\Gamma u_s g) \mathrm{d}s \mathrm{d}Ug \\
&\qquad = \frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{\left\{Ug; e^{\frac{t_g}{2}} < r\right\}} \int_0^1 f(\Gamma u_s g) \mathrm{d}s \mathrm{d}Ug \\
&\qquad = \frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{\left\{Ug; r_\varepsilon \le e^{\frac{t_g}{2}} < r\right\}} \int_0^1 f(\Gamma u_s g) \mathrm{d}s \mathrm{d}Ug \\
&\qquad\quad + \frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{\left\{Ug; e^{\frac{t_g}{2}} < r_\varepsilon\right\}} \int_0^1 f(\Gamma u_s g) \mathrm{d}s \mathrm{d}Ug.
\end{aligned}
$$

By choice of r_ε, the first integral satisfies

$$
\begin{aligned}
&\frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{\left\{Ug; r_\varepsilon \le e^{\frac{t_g}{2}} < r\right\}} \int_0^1 f(\Gamma u_s g) \mathrm{d}s \mathrm{d}Ug \\
&\qquad \approx \frac{\mathrm{vol}\big(B_r(0)\big) - \mathrm{vol}\big(B_{r_\varepsilon}(0)\big)}{\mathrm{vol}\big(B_r(0)\big)} \left(\frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int_{\Gamma/G} f(x) \mathrm{d}x + \varepsilon \right) \\
&\qquad \overset{r\to\infty}{\longrightarrow} \frac{\mathrm{vol}(\Gamma_\infty\backslash U)}{\mathrm{vol}(\Gamma\backslash G)} \left(\int_{\Gamma/G} f(x) \mathrm{d}x + \varepsilon \right).
\end{aligned}
$$

The second integral is bounded by

$$
\left| \frac{1}{\mathrm{vol}\big(B_r(0)\big)} \int_{\left\{Ug; e^{\frac{t_g}{2}} < r_\varepsilon\right\}} \int_0^1 f(\Gamma u_s g) \mathrm{d}s \mathrm{d}Ug \right| \le \frac{\mathrm{vol}\big(B_{r_\varepsilon}(0)\big)}{\mathrm{vol}\big(B_r(0)\big)} \|f\|_\infty \overset{r\to\infty}{\longrightarrow} 0.
$$

As ε was arbitrary, this proves the claim. □

It remains to derive a counting statement from the averaged counting obtained above. Note the upcoming choice of the radii which corresponds to the requirement that the "balls" under consideration ought to be *well-rounded*.

Proposition 4. *Given $r \in \mathbb{R}$, we define $B_r = B_{e^r}(0)$. Then*

$$
\frac{1}{\mathrm{vol}(B_r)} |\Gamma \cdot e_2 \cap B_r| \overset{r\to\infty}{\longrightarrow} \frac{\mathrm{vol}(\Gamma_\infty\backslash U)}{\mathrm{vol}(\Gamma\backslash G)} = \frac{3}{\pi^2}.
$$

Proof. Note that $\Gamma \cdot e_2 = e_2\Gamma$ by definition of the action $G \curvearrowright \mathbb{R}^2$. Let $\varepsilon > 0$ be arbitrary and choose $\delta > 0$ such that $\frac{\mathrm{vol}(B_{r+\delta})}{\mathrm{vol}(B_r)} < 1 + \varepsilon$ for all $r \ge 1$. Indeed $\mathrm{vol}(B_{r+\delta}) = \mathrm{vol}(B_r)e^{2\delta}$, so that any $\delta > 0$ satisfying $\delta < \frac{\log(1+\varepsilon)}{2}$ will do. There is some symmetric, open neighbourhood $V \subseteq G$ of the identity, such that $B_r V \subseteq B_{r+\delta}$

for all $r \geq 1$. To this end we use the discussion of the KAK-decomposition for $\mathrm{SL}_2(\mathbb{R})$ and choose as an open neighbourhood the set

$$V = K\left\{a_t; |t| < 2\delta\right\}K.$$

For any $g = ka_tl \in V$ one calculates

$$\|vg\| = \|vka_t\| \leq e^{\frac{|t|}{2}}\|v\| < e^{\delta}\|v\|.$$

Hence V has the desired properties.

Now let $g \in V$ arbitrary, then

$$\begin{aligned} F_{r+\delta}(g\Gamma) &= \frac{1}{\mathrm{vol}(B_{r+\delta})}|e_2\Gamma g \cap B_{r+\delta}| = \frac{1}{\mathrm{vol}(B_{r+\delta})}|e_2\Gamma \cap B_{r+\delta}g^{-1}| \\ &\geq \frac{1}{\mathrm{vol}(B_{r+\delta})}|e_2\Gamma \cap B_r| > \frac{1}{1+\varepsilon}\frac{1}{\mathrm{vol}(B_r)}|e_2\Gamma \cap B_r| \\ &= \frac{1}{1+\varepsilon}F_r(\Gamma) \end{aligned}$$

If now $\varphi \in C_c(\Gamma\backslash G)$ is non-negative, has integral 1, support contained in ΓV (which is an open neighbourhood of Γ) and does not vanish at Γ, then the above implies

$$F_r(\Gamma) \leq (1+\varepsilon)\int_{\Gamma/G} F_{r+\delta}(x)\varphi(x)\mathrm{d}x \overset{r\to\infty}{\longrightarrow} (1+\varepsilon)\frac{\mathrm{vol}(\Gamma_\infty\backslash U)}{\mathrm{vol}(\Gamma\backslash G)},$$

and hence we obtain

$$\limsup_{r\to\infty} \frac{1}{\mathrm{vol}(B_r)}|e_2\Gamma \cap B_r| \leq \frac{\mathrm{vol}(\Gamma_\infty\backslash U)}{\mathrm{vol}(\Gamma\backslash G)}.$$

On the other hand the same argument yields that for all $g \in V$ and $r > 1+\delta$ we have

$$\begin{aligned} F_r(\Gamma) &= \frac{1}{\mathrm{vol}(B_r)}|e_2\Gamma \cap B_r| \geq \frac{1}{\mathrm{vol}(B_r)}|e_2\Gamma \cap B_{r-\delta}g^{-1}| \\ &= \frac{1}{\mathrm{vol}(B_r)}|e_2\Gamma g \cap B_{r-\delta}| > \frac{1}{1+\varepsilon}F_{r-\delta}(\Gamma g) \end{aligned}$$

and thus

$$F_r(\Gamma) \geq \frac{1}{1+\varepsilon}\int_{\Gamma/G} F_{r-\delta}(x)\varphi(x)\mathrm{d}x \overset{r\to\infty}{\longrightarrow} \frac{1}{1+\varepsilon}\frac{\mathrm{vol}(\Gamma_\infty\backslash U)}{\mathrm{vol}(\Gamma\backslash G)}$$

and as ε was arbitrary, the claim follows. □

Lemma 6. *For our choice of normalization of the Haar measure, we have*

$$\mathrm{vol}({}_{\Gamma}\backslash^{G}) = \frac{\pi^2}{3}.$$

Proof. We combine two disintegration formulae. First we recall that $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ by Moebius transformations with $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\mathbf{i}) = \mathrm{SO}_2(\mathbb{R})$. Furthermore the hyperbolic area measure $\mathrm{d}m_{\mathbb{H}} = \frac{1}{y^2}\mathrm{d}x\mathrm{d}y$ is preserved by the action of $\mathrm{SL}_2(\mathbb{R})$. As discussed in an earlier talk, the functional given by sending $f \in C_c(G)$ to the number

$$\Lambda_1(f) = \int_{\mathbb{H}}\int_{\mathrm{SO}_2(\mathbb{R})} f(gk)\mathrm{d}m_{\mathrm{SO}_2(\mathbb{R})}(k)\mathrm{d}m_{\mathbb{H}}(g\cdot\mathbf{i})$$

defines a Haar measure on $\mathrm{SL}_2(\mathbb{R})$. In particular, uniqueness of the Haar measure up to multiplicative constants implies that there is some $C > 0$ such that for all $f \in C_c(G)$ holds

$$\Lambda(f_1) = C\int_{U/G}\int_U f(ug)\mathrm{d}m_U(u)\mathrm{d}m_{\mathbb{R}^2}(Ug).$$

The above equality applies to simple functions as is usually discussed in measure theory.

$$\begin{tikzcd}[column sep=small] & G \arrow[dl] \arrow[dr] & \\ G/K & & U\backslash G \end{tikzcd}$$

We will fix a fundamental domain F for Γ/G and calculate $\Lambda_1(\mathbb{1}_F)$, which will be relatively easy. Next we fix some appropriate function $f \in L^1(G)$ and calculate both $\Lambda_1(f)$ and

$$\Lambda_2(f) = \int_{U/G} \int_U f(ug) \mathrm{d}u \mathrm{d}Ug$$

for some function whose Haar integral does not vanish. In particular, $\Lambda_1(f)\Lambda_2(f) \neq 0$ and thus $C = \frac{\Lambda_1(f)}{\Lambda_2(f)}$. In particular we obtain that for our choice of the Haar measure holds

$$\mathrm{vol}({}_\Gamma\backslash^G) = \frac{\Lambda_2(f)}{\Lambda_1(f)} \Lambda_1(\mathbb{1}_F).$$

First of all, we recall that the set

$$\begin{aligned} F &= T^1 \left\{ z \in \mathbb{H}; |z| \geq 1, \Re(z) \in [-\tfrac{1}{2}, \tfrac{1}{2}] \right\} \\ &= \left\{ u_s a_t k; s \in [-\tfrac{1}{2}, \tfrac{1}{2}], e^{-t} \geq \sqrt{1 - s^2} \right\} \\ &= \left\{ u_s a_t k; t \in (-\infty, 0], s \in [-\tfrac{1}{2}, \tfrac{1}{2}] \right\} \\ &\quad \sqcup \left\{ u_s a_t k; t \in [0, -\log(\tfrac{\sqrt{3}}{2})), |s| \in [\sqrt{1 - e^{-t}}, \tfrac{1}{2}] \right\} \end{aligned}$$

is (up to a set of zero measure) a fundamental domain for $\Gamma\backslash G$.

Claim 1. *The map*

$$g\mathrm{SO}_2(\mathbb{R}) = z \mapsto \int_{\mathrm{SO}_2(\mathbb{R})} \mathbb{1}_F(gk) \mathrm{d}m_K(k)$$

is identiacal to 2π *times the indicator function of* $\mathcal{F} = \left\{ z \in \mathbb{H}; |z| \geq 1, \Re(z) \in [-\frac{1}{2}, \frac{1}{2}] \right\}$.

Indeed, by definition the set F is invariant under $\mathrm{SO}_2(\mathbb{R})$ on the right. Hence $\mathbb{1}_F(gk) = \mathbb{1}_F(g)$ and thus the claim follows easily. Hence in order to calculate the volume of $\Gamma\backslash G$ with respect to the measure induced by Λ_1, it suffices to calculate $m_{\mathbb{H}}(\mathcal{F})$. One obtains

$$m_{\mathbb{H}}(\mathcal{F}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} \mathrm{d}y \mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^2}} \mathrm{d}x = 2 \arcsin(\tfrac{1}{2})$$

and thus $\Lambda_1(F) = 4\pi \arcsin(\frac{1}{2})$.

On the other hand, we know that the set

$$\begin{aligned} E &= T^1 \left\{ z \in \mathbb{H}; \Re(z) \in [0, \tfrac{1}{2}], \Im(z) \geq 1 \right\} \\ &\equiv \left\{ u_s a_t k; s \in [0, \tfrac{1}{2}], t \leq 0 \right\} \end{aligned}$$

is injective and contains an open subset of $\mathrm{SL}_2(\mathbb{R})$. We calculate the measure of E in both ways. First of all, one obtains

$$\Lambda_1(\mathbb{1}_E) = 2\pi \int_0^{\frac{1}{2}} \int_1^{\infty} \frac{1}{y^2} \mathrm{d}y \mathrm{d}x = \pi,$$

and on the other hand one obtains

$$\Lambda_2(\mathbb{1}_{E^{-1}}) = \frac{1}{2} \mathrm{vol}(B_1(0)) = \frac{\pi}{2},$$

so that we obtain $C = 2$. Next we note that

$$\arcsin(\tfrac{1}{2}) = \arcsin(\sin(\tfrac{\pi}{6})) = \tfrac{\pi}{6},$$

and thus follows

$$\mathrm{vol}({}_{\Gamma}\backslash^{G}) = \Lambda_2(\mathbb{1}_F) = \frac{1}{2}\Lambda_1(\mathbb{1}_F) = \frac{\pi^2}{3} = 2\zeta(2).$$

$\square$

Email address: `manuel.luethi@math.ethz.ch`