Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
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Functional Analysis I

D-MATH

Student ID: 00-000-000

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Important Information

1. Exam duration: 3 hours. Put your student card or another document on the table.
2. No aiding material or electronic devices are allowed during the exam.
3. Use only pens with black or blue permanent ink. Do not use other colors, pens with erasable ink or pencils. **Do not use ink correctors (Tipp-Ex).**
4. Use a new, blank sheet for each different problem and indicate which problem you are solving. Write your **student ID** and the **Reference Number** on every sheet.
5. Sort the sheets according to the problem number before they are collected.
6. Write clearly! What is not clearly readable will be ignored.
7. You may invoke the main results seen in the course (e.g. Hahn-Banach, the Closed Graph Theorem), their main corollaries and the results quoted in the **Compendium** given at the end of this envelope. Aside from this, every step in your argument must be proved.
8. Note: In order to obtain the top grade, it is not necessary to solve all problems.

Please do not fill the table!

Ex.	Points	Check
1	[10]	
2	[5]	
3	[5]	
4	[10]	
5	[10]	
6	[10]	
Total	[50]	

Problem 1 [10 Points]

Let (M, d) be a nonempty metric space.

(i) When is a set $\Omega \subset M$ of second Baire category?

(ii) Suppose M is complete. By Baire's theorem, what property does M have?

(iii) Suppose now that M is countable and that no point in M is isolated. (Recall that, if X is a topological space, a point $x \in X$ is *isolated* if $\{x\}$ is open.) Prove that M is not complete.

(iv) Does there exist a compact, countable, infinite and complete metric space? If so, find an example, if not, explain why.

(v) Prove that $\mathbb{R}$ cannot be written as a countable union of closed, bounded, pairwise disjoint intervals.

Hint: Suppose that the intervals $[a_i, b_i]$ with $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, $i \in \mathbb{N}$, cover $\mathbb{R}$ and are mutually disjoint. Which properties does the set $\bigcup_{i=1}^{\infty}\{a_i, b_i\}$ have?

Problem 2 [5 Points]

Let $T : \ell^1 \to \ell^2$ be given by $Tx = x$. Consider the space $X = \ell^1 \times \ell^2$ and the subspaces

$$U = \ell^1 \times \{0\}, \qquad V = \Gamma_T \quad \text{(the graph of } T\text{)}, \qquad W = U + V.$$

Which of these spaces is closed, which is not closed, and why?

Problem 3 [5 Points]

Recall that a sequence $(b_n)_{n\in\mathbb{N}}$ in a Banach space X is a *Schauder basis* for X if any $x \in X$ can be uniquely represented as a convergent series $x = \sum_{n=1}^{\infty} x_n b_n$ with coefficients $x_n \in \mathbb{R}$.

(i) Show that the unit vectors

$$e_n = (0, 0, \ldots, 0, \underbrace{1}_{n^{\text{th}} \text{ pos.}}, 0, \ldots), \quad n \in \mathbb{N},$$

define a Schauder basis for any ℓ^p, $p \in [1, \infty[$ and also for the space

$$c_0 = \left\{ x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty \; ; \; \lim_{n\to\infty} x_n = 0 \right\}.$$

(ii) Is $(e_n)_{n\in\mathbb{N}}$ a Schauder basis for ℓ^∞?

Problem 4 [10 Points]

(i) State and prove the Closed Graph Theorem.

(ii) Let X, Y, Z be Banach spaces, let $A : X \to Y$ be a linear map and let $B \in L(Y, Z)$ (i.e. B is linear and continuous) and injective. Prove that if $BA \in L(X, Z)$, then $A \in L(X, Y)$.

Problem 5 [10 Points]

(i) State Arzelà-Ascoli's Theorem, defining explicitly the notion of "equicontinuity".

(ii) Let $0 < \alpha \leq 1$ and let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded subset. For continuous functions $\varphi : \Omega \to \mathbb{R}$, consider the so-called *Hölder norm*

$$\|\varphi\|_{C^{0,\alpha}(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha},$$

and the corresponding normed space:

$$C^{0,\alpha}(\overline{\Omega}, \mathbb{R}) = \left\{\varphi \in C^0(\overline{\Omega}, \mathbb{R}) \; ; \; \|\varphi\|_{C^{0,\alpha}(\Omega)} < \infty\right\}.$$

Prove that a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ which is bounded with respect to the Hölder norm has a uniformly convergent subsequence.

(iii) Let $X = L^2(]0,1[, \mathbb{R})$ and let $T : X \to X$ be given by $T(f)(x) = \int_0^x f(y)dy$.

(a) Prove that $T(X) \subset C^{0,1/2}([0,1], \mathbb{R})$

(b) Use (ii) to prove that T is a compact operator from X to $Y = C^0([0,1], \mathbb{R})$.

Problem 6 [10 Points]

Let H be an infinite-dimensional Hilbert space and let $T \in L(H)$ be an injective, compact, self-adjoint operator.

(i) Show that 0 is in the spectrum of $\sigma(T)$.

(ii) Show that $\mathrm{Im}(T) = T(H)$ is dense in H but $\mathrm{Im}(T) \neq H$.

Compendium of Functional Analysis

1. *ℓ^p Spaces* For a sequence $x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, and $p \in [1,\infty]$, we set

$$\|x\|_{\ell^p} = \begin{cases} \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup_{n\in\mathbb{N}} |x_n| & \text{if } p = \infty. \end{cases}$$

   Then $\ell^p = \{x = (x_n)_{n\in\mathbb{N}} \,;\, \|x\|_{\ell^p} < \infty\}$ are Banach spaces.

   For $p \le q$ there holds $\|x\|_q \le \|x\|_p$ and we have the inclusion $\ell^p \subseteq \ell^q$, which is strict for $p \neq q$.

2. *Hölder's Inequality* If $\Omega \subseteq \mathbb{R}^n$ is measurable and $p, q \in [1,\infty]$ are so that $1/p + 1/q = 1$, then for every $f \in L^p(\Omega,\mathbb{R})$ and $g \in L^q(\Omega,\mathbb{R})$ there holds

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

3. *Open Mapping Theorem* Let X, Y be Banach spaces and let $L \in L(X,Y)$ be surjective. Then L is an open mapping. In particular, if L is bijective, then $L^{-1} \in L(Y,X)$.

4. *Hahn-Banach Theorem* Let X be a normed space, let $M \subset X$ be a closed linear subspace, $M \neq X$, and let $x_0 \in M$ so that

$$d = \operatorname{dist}(x_0, M) = \inf_{x\in M} \|x_0 - x\|_X > 0.$$

   Then there exists $l \in X^*$ so that $l|_M = 0$ and

$$\|l\|_{X^*} = 1, \quad l(x_0) = d.$$

5. *Compact Operators* Let X, Y be a Banach space. An operator $T \in L(X,Y)$ is called compact if $T(B_1(0;X))$ has compact closure in Y.

   Equivalently, $T \in L(X,Y)$ is compact if and only if it maps weakly to strongly converging sequences, i.e. for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \overset{w}{\to} x$ as $n \to \infty$ in X, there holds $Tx_n \to Tx$ in Y as $n \to \infty$.

6. *Spectrum* Let X be a Banach space over $\mathbb{C}$ and let $A : D_A \subseteq X \to X$ be linear. The Resolvent Set of A is

$$\rho(A) = \Big\{\lambda \in \mathbb{C} \,;\, \lambda \operatorname{Id} - A : D_A \to X \text{ is bijective and } (\lambda \operatorname{Id} - A)^{-1} \in L(X)\Big\},$$

   where $\operatorname{Id} : X \to X$ denotes the identity map.

   The Spectrum of A is $\sigma(A) = \mathbb{C} \setminus \rho(A)$.