Exercise 1.1 Consider the set of all real-valued sequences

$$S = \{(s_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N} : s_n \in \mathbb{R}\}.$$

Prove that the function $d : S \times S \to [0, \infty)$ defined by

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

is a metric over S, and that (S, d) is a complete metric space.

Hint: the function $t \mapsto \frac{t}{1+t}$, $t > 0$, is concave.

Exercise 1.2 Let $\Omega \subseteq \mathbb{R}^m$ be an open subset and let $(\Omega_n)_{n\in\mathbb{N}}$ be an exhaustion of Ω by open sets with compact closure, that is, each $\Omega_n \subseteq \mathbb{R}^m$ is open, $\overline{\Omega}_n$ is compact and contained in Ω, $\Omega_n \subseteq \Omega_{n+1}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$. Define

$$d(f, g) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{\|f - g\|_{C^0(\overline{\Omega}_n)}}{1 + \|f - g\|_{C^0(\overline{\Omega}_n)}}$$

for every continuous, real-valued functions $f, g \in C^0(\Omega, \mathbb{R})$.

(a) Prove that d defines a metric in $C^0(\Omega, \mathbb{R})$.

(b) Prove that $(C^0(\Omega, \mathbb{R}), d)$ is a complete metric space.

(c) Let $C^0_c(\Omega, \mathbb{R})$ be the set of continuous functions with compact support in Ω. Prove that $C^0_c(\Omega, \mathbb{R})$ is dense in $(C^0(\Omega, \mathbb{R}), d)$.

Remark. The topology defined by d is called topology of the convergence on compact subsets of Ω. It is possible to prove that it does not depend on the chosen exhaustion.

Exercise 1.3 Let (X, d) be a metric space. Prove that the following are equivalent:

(a) The complement of every meager set is dense in X.

(b) The interior of every meager set is empty.

(c) The empty set is the only open and meager set.

(d) Countable intersections of open dense sets are dense.

Hint: recall that A is dense in X if and only if its complement has empty interior.

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Remark. Thanks to Baire's Category Theorem, each of the above conditions are satisfied in a complete metric space.

Exercise 1.4 Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence. Define, for every $p \in [1,\infty]$,

$$\|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} = \begin{cases} \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup_{n\in\mathbb{N}} |x_n| & \text{if } p = \infty, \end{cases}$$

and let $\ell^p = \{(x_n)_{n\in\mathbb{N}} \mid \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} < \infty\}$. For every $p \in [1,\infty]$, $(\ell^p, \|\cdot\|_{\ell^p})$ is a Banach space.

Let now $1 \leq p < q \leq \infty$. Prove that:

(a) $\ell^p \subsetneq \ell^q$ and $\|(x_n)_{n\in\mathbb{N}}\|_{\ell^q} \leq \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p}$ for every $(x_n)_{n\in\mathbb{N}} \in \ell^p$.

(b) ℓ^p is meager in ℓ^q.

(c) $\bigcup_{1\leq p<q} \ell^p \subsetneq \ell^q$.

Hint for (b): The set $A_n = \{(x_n)_{n\in\mathbb{N}} \in \ell^q \mid \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} \leq n\} \subseteq \ell^q$ is closed in ℓ^q and has empty interior in ℓ^q.

Remark. Notice that $\ell^p = L^p(\mathbb{N}, \mathcal{A}, \mu)$, where $\mathcal{A}$ is the σ-algebra of all subsets of $\mathbb{N}$ and μ is the counting measure, i. e. $\mu(M)$ is the cardinality of M (possibly ∞).