Exercise 2.1 Let X be a vector space. An *algebraic basis* for X is a subset $E \subset X$ such that every $x \in X$ is uniquely given as *finite* linear combination of elements in E.

(a) Let $(X, \|\cdot\|)$ be a Banach space. Show that any algebraic basis for X is either finite or uncountable.

(b) Find an example of a normed space with a countably infinite algebraic basis.

Hint for (a): Assume that X has a countably infinite algebraic basis $\{e_1, e_2, \ldots\}$ and deduce a contradiction to Baire's Lemma by considering the sets $A_n = \operatorname{span}\{e_1, \ldots, e_n\}$.

Exercise 2.2 Let $f \in C^0([0,\infty))$ be a continuous function satisfying

$$\forall t \in [0,\infty): \lim_{n\to\infty} f(nt) = 0.$$

Prove that $\lim_{t\to\infty} f(t) = 0$.

Hint: Apply Baire's Lemma as in the proof of the uniform boundedness principle.

Exercise 2.3 Let

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N:\ x_n = 0\}$$

be the space of compactly supported sequences and

$$c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}.$$

be the space of sequences converging to zero.

(i) Show that $(c_c, \|\cdot\|_{\ell^\infty})$ is *not* complete. What is the completion of this space?

(ii) Prove the strict inclusion

$$\bigcup_{p\in[1,\infty)} \ell^p \subsetneq c_0.$$

Exercise 2.4 Show that the subspaces

$$U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}:\ x_{2n} = 0\},$$
$$V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}:\ x_{2n-1} = nx_{2n}\}$$

are both closed in $(\ell^1, \|\cdot\|_{\ell^1})$ while the subspace $U \oplus V$ is not closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

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Hint. Prove that if any sequence $(x_k)_{k\in\mathbb{N}}$ of elements $x_k = (x^{k,n})_{n\in\mathbb{N}} \in \ell^1$ converges to $(x_n)_{n\in\mathbb{N}}$ in ℓ^1 for $k \to \infty$, then each entry $x_{n,k}$ converges in $\mathbb{R}$ to x_n for $k \to \infty$. For the second part, show first that c_c (see Exercise 2.1) is a subset of $U \oplus V$.

Exercise 2.5 Let $(X, \|\cdot\|)$ be a normed vector space. Prove that the following statements are equivalent.

(i) $(X, \|\cdot\|)$ is a Banach space.

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\displaystyle\sum_{k=1}^{\infty} \|x_n\| < \infty$ the limit $\displaystyle\lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists.

Hint: A Cauchy sequence converges if and only if it has a convergent subsequence.