Exercise 7.1 Let $(H, (\cdot,\cdot)_H))$ be a Hilbert space. Let $Y \subset H$ be any subspace and let $f\colon Y \to \mathbb{R}$ be a continuous linear functional. By the Hahn-Banach Theorem there exists an extension $F\colon H \to \mathbb{R}$ with $F|_Y = f$ and $\|F\| = \|f\|$. Prove that F is unique.

Exercise 7.2 Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and $Q \subset H$ a nonempty convex subset. Let $x \in H$ with distance $d := \operatorname{dist}(x, Q)$ from Q. Prove that:

(i) Every sequence $(x_n)_{n\in\mathbb{N}}$ in Q with $\lim\limits_{n\to\infty} \|x_n - x\|_H = d$ is a Cauchy sequence in H.

(ii) If Q is closed in H, then there exists a *unique* $y \in Q$ with $\|x - y\|_H = d$.

Exercise 7.3 Let $(X, \|\cdot\|_X)$ be a normed space.

(i) Let A be a subset of X and let $\operatorname{conv}(A)$ denote its convex hull. Prove the following characterization:

$$\operatorname{conv}(A) = \left\{ \sum_{k=1}^{n} \lambda_k x_k \;\middle|\; n \in \mathbb{N},\ x_1, \ldots, x_n \in A,\ \lambda_1, \ldots, \lambda_n \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1 \right\}.$$

(ii) Let $A, B \subset X$ be compact, convex subsets. Prove that $\operatorname{conv}(A \cup B)$ is compact.

Exercise 7.4 *(Lions-Stampacchia).* Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space Let $a\colon H \times H \to \mathbb{R}$ be a bilinear map so that:

(a) $a(x, y) = a(y, x)$ for every $x, y \in H$,

(b) there exists $\Lambda > 0$ so that $|a(x, y)| \leq \Lambda \|x\|_H \|y\|_H$ for every $x, y \in H$,

(c) there exists $\lambda > 0$ so that $a(x, x) \geq \lambda \|x\|_H^2$ for every $x \in H$.

Let moreover $f\colon H \to \mathbb{R}$ be a continuous linear functional. Consider the map $J\colon H \to \mathbb{R}$ given by

$$J(x) = a(x, x) - 2f(x).$$

Prove that, for any $K \subset H$ be a nonempty closed, convex subset, there exists a *unique* $y_0 \in K$ such that *both* the following inequalities hold:

(i) $J(y_0) \leq J(y)$ for every $y \in K$,

(ii) $a(y_0, y - y_0) \geq f(y - y_0)$ for every $y \in K$.

Exercise 7.5 Consider the spaces

$$c_0 := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k = 0\Big\}, \qquad c := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k \text{ exists}\Big\}.$$

with norm $\|\cdot\|_{\ell^\infty}$.

(i) Is c a Banach space?

(ii) Show that the dual space of $(c_0, \|\cdot\|_{\ell^\infty})$ is *isometrically* isomorphic to $(\ell^1, \|\cdot\|_{\ell^1})$.

(iii) To which space is the dual space of $(c, \|\cdot\|_{\ell^\infty})$ isomorphic?

Hints to Exercises.

7.1 Start by considering the case where Y is a *closed* subspace, and use Riesz representation theorem. Then reduce the general case to this one by considering the closure of f.

7.2 Use the parallelogram identity.

7.3 For (ii), show that

$$\operatorname{conv}(A \cup B) = \bigcup_{\substack{s,t \geq 0 \\ s+t=1}} (sA + tB),$$

and then prove that the right-hand side is compact.

7.4 Note first that $a(\cdot,\cdot)$ is a scalar product topologically equivalent to $(\cdot,\cdot)_H$, then use Riesz Representation Theorem for f with this scalar product and Exercise 7.2.

7.5 For (ii), compare with Satz 4.4.1 (dual of L^p). For (iii): how can one transform a sequence in c into a sequence in c_0?