Exercise 9.1 Let ℓ^∞ be the space of real-valued bounded sequences and let c be the subspace of converging sequences. Consider the functional

$$\lim : c \to \mathbb{R} \quad \lim(x_n) = \lim_{n\to\infty} x_n.$$

(i) Prove that it extends to a continuous linear functional $\mathfrak{lim} : \ell^\infty \to \mathbb{R}$ with norm $\|\mathfrak{lim}\| = 1$ and that there holds

$$\liminf_{n\to\infty} x_n \leq \mathfrak{lim}(x_n) \leq \limsup_{n\to\infty} x_n.$$

(ii) Use such construction to prove that the space ℓ^1 is not reflexive.

Exercise 9.2 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $T\colon X \to Y$ be a linear operator. Prove that the following statements are equivalent.

(i) T is continuous.

(ii) T is weak-weak sequentially continuous, namely if $(x_n)_{n\in\mathbb{N}}$ is any weakly converging sequence X, then Tx_n is weakly convergent in Y.

Exercise 9.3 Let $(X, \|\cdot\|_X)$ be a finite-dimensional normed space. Prove that then strong and weak topologies coincide, namely that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is weakly convergent if and only if it is strongly convergent.

Exercise 9.4 Let $(H, \langle\cdot,\cdot\rangle)$ be a real vector space and let $(e_n)_{n\in\mathbb{N}} \subseteq X$ be an *orthonormal system for* H, that is a countable set of elements so that

$$\langle e_j, e_k\rangle = \delta_{jk} \quad \text{for every } j, k \in \mathbb{N}.$$

(i) Prove $e_n \xrightarrow{w} 0$ as $n \to \infty$.

(ii) Suppose now that $(e_n)_{n\in\mathbb{N}}$ forms a *Hilbert basis for* H, i.e. that $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is a dense subspace of H. Prove that for every $x \in H$ there holds

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n, \tag{1}$$

and that *Parseval's Identity* holds:

$$\|x\|_H = \left(\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2\right)^{1/2}. \tag{2}$$

Exercise 9.5 Let $(H, (\cdot,\cdot)_H)$ be a real Hilbert space.

(i) Prove that if the sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges weakly to x and $\|x_n\|_H \to \|x\|_H$, then it converges strongly to x.

(ii) Suppose $(x_n)_{n\in\mathbb{N}}$ converges weakly to x and $(y_n)_{n\in\mathbb{N}} \subseteq X$ converges strongly to y. Prove that $(x_n, y_n)_H \to (x, y)_H$.

(iii) Suppose $x \in H$ with $\|x\|_H \leq 1$, prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H satisfying $\|x_n\|_H = 1$ for all $n \in \mathbb{N}$ and $x_n \overset{\mathrm{w}}{\rightharpoonup} x$ as $n \to \infty$.

(iv) Prove the *Riemann-Lebesgue Lemma*: Let $f_n \colon [0, 2\pi] \to \mathbb{R}$ given by $f_n(t) = \sin(nt)$ for $n \in \mathbb{N}$, then $f_n \overset{\mathrm{w}}{\rightharpoonup} 0$ in $L^2((0, 2\pi), \mathbb{R})$ as $n \to \infty$.

Hints to Exercises.

9.1 For (i), one inequality follows from Hahn-Banach; for the other one argue by contradiction.

9.4 Use (after proving it) *Bessel's inequality*: $\sum_{n=0}^{\infty}|(x,e_n)_H|^2 \leq \|x\|_H^2$.

9.5 For (iii), use Exercise 9.4. Recall that in every Hilbert space the Graham-Schmidt process allows for construction of orthonormal systems.