Exercise 10.1 For each of the Banach spaces below (each one endowed with its standard norm), find a sequence which is bounded but does not have a convergent subsequence:

(i) $L^p((0,1),\mathbb{R})$ for $1 \leq p \leq \infty$;

(ii) $c_0 \subset \ell^\infty$, the space of sequences converging to zero.

Exercise 10.2 Prove that the following statements are equivalent.

(i) $(X, \|\cdot\|_X)$ is separable.

(ii) $B = \{x \in X \mid \|x\|_X \leq 1\}$ is separable.

(iii) $S = \{x \in X \mid \|x\|_X = 1\}$ is separable.

Exercise 10.3 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. Recall that if $T \in L(X,Y)$, then its dual operator T^* is in $L(Y^*, X^*)$ and it is characterised by the property

$$\langle T^* y^*, x\rangle_{X^* \times X} = \langle y^*, Tx\rangle_{Y^* \times Y} \quad \text{for every } x \in X \text{ and } y^* \in Y* .$$

Prove the following facts about dual operators.

(i) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$.

(ii) If $T \in L(X,Y)$ and $S \in L(Y,Z)$, then $(S \circ T)^* = T^* \circ S^*$.

(iii) If $T \in L(X,Y)$ is bijective with inverse $T^{-1} \in L(Y,X)$, then $(T^*)^{-1} = (T^{-1})^*$.

(iv) Let $\mathcal{I}_X \colon X \hookrightarrow X^{**}$ and $\mathcal{I}_Y : Y \hookrightarrow Y^{**}$ be the canonical inclusions. Then,

$$\forall\, T \in L(X,Y): \quad \mathcal{I}_Y \circ T = (T^*)^* \circ \mathcal{I}_X.$$

Exercise 10.4 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T \in L(X,Y)$. Prove the following.

(i) If T is an isomorphism with $T^{-1} \in L(Y,X)$, then T^* is an isomorphism.

(ii) If T is an isometric isomorphism, then T^* is an isometric isomorphism.

(iii) If X and Y are both reflexive, then the reverse implications of i and ii hold.

(iv) If $(X, \|\cdot\|_X)$ is a reflexive Banach space isomorphic to the normed space $(Y, \|\cdot\|_Y)$, then Y is reflexive.

Exercise 10.5 Let let $\Omega \subset \mathbb{R}^m$ be an open, bounded subset. For fixed $g \in L^2(\mathbb{R}^m)$, we define the map $V\colon L^2(\Omega) \to \mathbb{R}$ by

$$V(f) = \int_\Omega \int_\Omega g(x-y) f(y) f(x) \, dy \, dx,$$

and for fixed $h \in L^2(\Omega)$ we define the map $E\colon L^2(\Omega) \to \mathbb{R}$ by

$$E(f) = \|f-h\|^2_{L^2(\Omega)} + V(f).$$

(i) Check that V is well-defined, namely that the integral is absolutely convergent for every f and g.

(ii) Prove that V is weakly sequentially continuous.

(iii) Under the assumption $g \geq 0$ almost everywhere, prove that E restricted to

$$L^2_+(\Omega) := \{f \in L^2(\Omega) \mid f(x) \geq 0 \text{ for almost every } x \in \Omega\}$$

attains a global minimum.

Hints to Exercises.

10.2 Use Satz 3.4.1.

10.4 Use Exercise 10.3.

10.5 Begin by considering the convolution operator $f \mapsto g * f = \int_\Omega g(x-y)f(y)dy$ from $L^2(\Omega)$ into itself. Prove that is maps weakly convergent sequences into strongly converging subsequences