Exercise 11.1 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We denote by

$$K(X,Y) = \{T \in L(X,Y) \mid \overline{T(B_1(0))} \subset Y \text{ compact}\}$$

the set of compact operators between X and Y. Prove the following statements.

(i) $T \in L(X,Y)$ is a compact operator if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in Y.

(ii) If $(Y, \|\cdot\|_Y)$ is complete, then $K(X,Y)$ is a closed subspace of $L(X,Y)$.

(iii) Let $T \in L(X,Y)$. If its range $T(X) \subset Y$ is finite-dimensional, then $T \in K(X,Y)$.

(iv) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If T or S is a compact operator, then $S \circ T$ is a compact operator.

(v) If X is reflexive, then any operator $T \in L(X,Y)$ which maps weakly convergent sequences to strongly convergent sequences, that is

$$x_n \overset{w}{\rightharpoonup} x \text{ in } X \quad \Longrightarrow \quad Tx_n \to x \text{ in } Y,$$

is a compact operator.

Exercise 11.2 Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C^1([0,1],\mathbb{R})$ so that, for every $n \in \mathbb{N}$,

$$f_n(0) = a \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|f_n'\|_{L^\infty((0,1))} \leq C,$$

for some a and C. Show that $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

Exercise 11.3 Let $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^m$ be a bounded open subset. Given $k \in L^2(\Omega \times \Omega, \mathbb{C})$, consider the linear operator $K \colon L^2(\Omega) \to L^2(\Omega, \mathbb{C})$ defined by

$$(Kf)(x) = \int_\Omega k(x,y) f(y)\, dy$$

(i) Prove that K is well-defined, i. e. $Kf \in L^2(\Omega, \mathbb{C})$ for any $f \in L^2(\Omega, \mathbb{C})$.

(ii) Prove that K is a compact operator.

(iii) Find an explicit expression for the adjoint $K^* : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ (recall that for complex-valued function, the scalar product in $L^2(\Omega, \mathbb{C})$ is $(f,g)_{L^2} = \int_\Omega f\overline{g} dx$).

Exercise 11.4 Let $\ell^p_{\mathbb{C}}$ denote the space of $\mathbb{C}$-valued sequences of summable p-th power, namely

$$\ell^p_{\mathbb{C}} := \Big\{ x : \mathbb{N} \to \mathbb{C} \ : \ \sum_{n\in\mathbb{N}} |x_n|^p < \infty \Big\},$$

where as usual we write $x_n = x(n)$. The space is endowed with its standard Banach norm $\|\cdot\|_{\ell^p_{\mathbb{C}}}$. Given $a \in \ell^\infty_{\mathbb{C}}$ we define the operator $T\colon \ell^2_{\mathbb{C}} \to \ell^2_{\mathbb{C}}$ by $(Tx)_n = a_n x_n$.

(i) Prove that $T \in L(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ and compute its operator norm.

(ii) Prove that T is self-adjoint if and only if $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

(iii) Prove that T is compact if and only if $\lim_{n\to\infty} a_n = 0$.

Hints to Exercises.

11.1 For (v), use Eberlein-Šmulian's Theorem.

11.3 Use repeatedly the theorem of Fubini-Tonelli. For (ii) Use Exercise 10.1 (v).