Some information on the Exam

Here is a list with useful information on the exam. If you have any question, please mail the course organizer.

- The exam will take place on Monday 27th January 2020, and it will be 3 hours long. For the time and place, please check mystudies in due time.
- The exam will consists in questions concerning the material presented in the lectures and on the exercise sheets. You will be asked to give definitions and statements seen in the course and to solve some exercises. Some of the exercises will ask for (reasonably short) proofs already seen in the lecture or in the exercise sheets.
- No aiding material is allowed during the exam. This includes: lecture notes, personal notes, exercise sheets, mobile phones or tablets, calculators. A **compendium** listing some theorems and useful results will be given with the exam.
- While solving the exercises, you may invoke the main results seen in the course (e.g. Hahn-Banach, the Closed Graph Theorem), their main corollaries and the results quoted in the compendium. Aside from this, every step in your argument must be proved.
- You may bring your own blank paper (A4 format).
- In order to obtain the maximal grade, it will not be necessary to solve all problems.

This Exercise sheet is thought as a mock exam. We recommend to approcach it after a thorough study of the whole course material, Exercise Sheets 1 to 12 included. The exercises are taken, with some adjustments, from past exams.

As in a regular exam, you may want to:

- Avoid the use any aid material such as notes, textbooks or friends (but you may consult the compendium given on the last page).
- Face all exercises without interruption,
- Keep track of the time.

The solution will be released, as usual, after some time. If you have any question you may always write to your assistant or to the course organizer.

Exercise 13.1 Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{C}$ and let $T \in L(X)$.

(i) Give the definition of spectral radius, of resolvent and of spectrum of T.

(ii) Assume that T satisfies $T^2 = T$. Compute the spectral radius and the spectrum of T.

Exercise 13.2 Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed spaces.

(i) Give the definition of weakly convergent sequence in X and prove that every weakly convergent sequence is bounded.

(ii) Let $L : X \to Y$ be a linear operator. Prove that L is continuous if and only if it is weak-weak sequentially continuous, namely if and only if, for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \overset{w}{\rightharpoonup} x$ in X, there holds $Lx_n \overset{w}{\rightharpoonup} Lx$ in Y.

(iii) Let $T \in L(X, Y)$. Use (ii) to prove that, if X is reflexive, $T\left(\overline{B_1(0)}\right)$ is closed.

Exercise 13.3 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces.

(i) Prove the Banach-Steinhaus theorem: if $(T_\lambda)_{\lambda\in\Lambda}$ is a family of linear, continuous operators from X to Y so that, for every fixed $x \in X$, there holds

$$\sup_{\lambda\in\Lambda} \|T_\lambda x\|_Y < \infty.$$

then

$$\sup_{\lambda\in\Lambda} \|T_\lambda\|_{L(X,Y)} < \infty.$$

Note: You are required to prove this result without resorting to the more general Uniform Boundedness Principle seen in the course.

(ii) Let X^* be the dual of X. Let $S \subseteq X$ be a subset of X so that, for every $x^* \in X^*$,

$$x^*(S) = \{x^*(x) : x \in S\}$$

is a bounded set in $\mathbb{R}$. Prove that then S is bounded.

Exercise 13.4 Let $(X, \|\cdot\|_X)$ be a normed space.

(i) Give the definition of weak sequentially closed subset $A \subseteq X$. Which of the following assertion is true (give a proof), which is false (give a counterexample)?

(a) A is weakly sequentially closed $\Longrightarrow$ A is closed.

(b) A is closed $\Longrightarrow$ A is weakly sequentially closed.

(ii) Let X, Y be Banach and let $T : X \to Y$ be a linear map so that

$$y^* \circ T \in X^* \quad \text{for every } y \in Y^*,$$

where X^*, Y^* denote the duals of X and Y respectively. Prove that the graph of T is weakly closed.

(iii) Let X, Y be Banach and let $T \in L(X, Y)$ be a linear and continuous map. Prove that $\ker(T)$ is weakly sequentially closed.

Exercise 13.5

(i) State the Principle of Calculus of Variations ("Variationsprinzip").

Let now $(H, \langle\cdot,\cdot\rangle)$ be a (nonempty) Hilbert space, let $p \in (1, \infty)$ be fixed and define the function

$$F : H \to \mathbb{R}, \qquad F(u) = \frac{\|u\|_H^p}{p}.$$

(ii) Prove that, for every fixed $v \in H$, the quantity

$$G(v) = \sup_{u \in H} \Big(\langle u, v\rangle - F(u)\Big)$$

is finite and there always exists some $u \in H$ that attains the supremum on the right-hand side.

(iii) Find an explicit expression for G involving only p and the norm of v.

Exercise 13.6 Let X, Y be Banach spaces and let $T \in L(X, Y)$ be a linear continuous operator with closed image.

(i) Prove that every topologically complemented (see the Compendium) subspace is closed.

(ii) Prove the equivalence of the following statements:

(a) $\mathrm{im}(T)$ and $\ker(T)$ are topologically complemented,

(b) There exists $S \in L(Y, X)$ with

$$STS = S \quad \text{and} \quad TST = T.$$

A Compendium of Functional Analysis

1. *Topological complements* A linear subspace of a normed space $V \subset X$ is complemented if and only if there exists a linear, continuous map $P : X \to X$ so that $P(X) = V$ and $P^2 = P$.

2. *Baire Theorem* Let (M, d) be a complete metric space. If $(D_n)_{n\in\mathbb{N}}$ is a sequence of closed subsets of M so that $\bigcup_{n\in\mathbb{N}} D_n$ has nonempty interior, then at least one of the D_n's has nontempty interior.

3. *Open Mapping Theorem* Let X, Y be Banach spaces and let $L \in L(X, Y)$ be surjective. Then L is an open mapping. In particular, if L is bijective, then $L^{-1} \in L(Y, X)$.

4. *Hahn-Banach Theorem* Let X be a (real) normed space, let $V \subset X$ be a linear subspace and let $L : V \to \mathbb{R}$ be linear with $L(x) \leq p(x)\ \forall x \in V$, where $p : X \to \mathbb{R}$ is a sublinear functional. Then there exists an extension of L, $L_{\text{ext}} : X \to \mathbb{R}$ so that $L_{\text{ext}}(x) \leq p(x)\ \forall x \in X$.

5. *Norms and Duality* Let X be a Banach space and let X^* be its dual. Then for every $x \in X$ there holds $\|x\|_X = \sup\{|x^*(x)| : x \in X^*, \|x^*\|_{X^*} \leq 1\}$.

6. *Eberlein-Šmulian Theorem* Let X be a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence. Then there exists a subsequence $(x_n)_{n\in\Lambda}$, $\Lambda \subset \mathbb{N}$ which is weakly convergent: $x_n \overset{w}{\rightharpoonup} x$ as $n \to \infty$, $n \in \Lambda$.

7. *Weak s.l.s.c. of the Norm* Let X be a Banach space and and let $(x_n)_{n\in\mathbb{N}} \subset X$ be so that $x_n \overset{w}{\rightharpoonup} x$. Then $\|x\|_X \leq \liminf_{n\to\infty} \|x_n\|_X$.