Exercise 1.1 Consider the set of all real-valued sequences

$$S = \{(s_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N} : s_n \in \mathbb{R}\}.$$

Prove that the function $d : S \times S \to [0, \infty)$ defined by

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

is a metric over S, and that (S, d) is a complete metric space.

Hint: the function $t \mapsto \frac{t}{1+t}$, $t > 0$, is concave.

Solution. Note first of all that for every $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in S$ there holds:

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \leq \sum_{n\in\mathbb{N}} 2^{-n} < \infty.$$

So d takes values in $[0, \infty)$. Clearly, d is symmetric, non-negative and vanishes only when $(x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$. We prove that the triangle inequality holds. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in S$. Since for every $n \in \mathbb{N}$ there holds:

$$\begin{aligned}
\frac{|x_n - z_n|}{1 + |x_n - z_n|} &= 1 - \frac{1}{1 + |x_n - z_n|} \\
&\leq 1 - \frac{1}{1 + |x_n - y_n| + |y_n - z_n|} \\
&= \frac{|x_n - y_n| + |y_n - z_n|}{1 + |x_n - y_n| + |y_n - z_n|} \\
&= \frac{|x_n - y_n|}{1 + |x_n - y_n| + |y_n - z_n|} + \frac{|y_n - z_n|}{1 + |x_n - y_n| + |y_n - z_n|} \\
&\leq \frac{|x_n - y_n|}{1 + |x_n - y_n|} + \frac{|y_n - z_n|}{1 + |y_n - z_n|},
\end{aligned}$$

this implies:

$$d((x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}) \leq d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) + d((y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}),$$

and thus d is a metric on S.

Last modified: 7 October 2019

We prove that the space (S,d) is complete. Note first that a Cauchy sequence in S is a sequence of sequences $a_k=(a_{k,n})_{n\in\mathbb{N}}$ so that, for every $\varepsilon>0$, there exists $K\in\mathbb{N}$ so that

$$d(a_k,a_l)=\sum_{n\in\mathbb{N}}2^{-n}\frac{|a_{k,n}-a_{l,n}|}{1+|a_{k,n}-b_{l,n}|}<\varepsilon\quad\text{for every } k,l\geq K.$$

Claim: $(a_k)_{k\in\mathbb{N}}$ *is a Cauchy sequence in* S *if and only if, for every fixed* $n\in\mathbb{N}$, $(a_{k,n})_{k\in\mathbb{N}}$ *is a Cauchy sequence in* $\mathbb{R}$.

Proof. Sufficiency: let $n\in\mathbb{N}$ and $\varepsilon>0$ be fixed and let $K\in\mathbb{N}$ be so that

$$d(a_k,a_l)=\sum_{n\in\mathbb{N}}2^{-n}\frac{|a_{k,n}-a_{l,n}|}{1+|a_{k,n}-a_{l,n}|}<\varepsilon\quad\text{for every } k,l\geq K.$$

Consequently, it follows that for every $n\in\mathbb{N}$

$$\frac{|a_{k,n}-a_{l,n}|}{1+|a_{k,n}-a_{l,n}|}<2^n\varepsilon,$$

so assuming without loss of generality that $\varepsilon<2^{-n}$, we deduce

$$|a_{k,n}-a_{l,n}|<\frac{2^n}{1-2^n\varepsilon}\varepsilon,$$

which implies that $(a_{k,n})_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$.

Necessity: note that, for every $k,l,N\in\mathbb{N}$ we may always estimate

$$\begin{aligned}\sum_{n\in\mathbb{N}}2^{-n}\frac{|a_{k,n}-a_{l,n}|}{1+|a_{k,n}-a_{l,n}|}&=\sum_{n=0}^{N}2^{-n}\frac{|a_{k,n}-a_{l,n}|}{1+|a_{k,n}-a_{l,n}|}+\sum_{n\geq N+1}2^{-n}\frac{|a_{k,n}-a_{l,n}|}{1+|a_{k,n}-a_{l,n}|}\\&\leq\sum_{n=0}^{N}2^{-n}\frac{|a_{k,n}-a_{l,n}|}{1+|a_{k,n}-a_{l,n}|}+2^{-N}.\end{aligned}$$

By assumption, for every $\varepsilon>0$ and every $n\in\mathbb{N}$, there exists $N(\varepsilon,n)\in\mathbb{N}$ so that $|a_{n,k}-a_{n,l}|\leq\varepsilon$ for every $k,l\geq N(\varepsilon,n)$. Consequently, we fix fist $M\in\mathbb{N}$ so that $2^{-M}\leq\varepsilon$ and then we choose $N(\varepsilon)$ to be the greatest element in the set $\{M,N(\varepsilon,1),\ldots N(\varepsilon,M)\}$ to deduce that $d(a_k,a_l)\leq 3\varepsilon$, which implies that $(a_k)_{k\in\mathbb{N}}$ is Cauchy in S. $\square$

Let now $(a_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in S. By Claim 1, for every n, $(a_{k,n})_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ and thus is convergent to some element $a_k\in\mathbb{R}$. Defining $a=(a_k)_{k\in\mathbb{N}}$ as an element of S and arguing as in the proof of the Claim above ("necessity" part), it is possible to find, for every $\varepsilon>0$, some $N(\varepsilon)\in\mathbb{N}$ so that $d(a_k,a)\leq\varepsilon$ for every $k\geq N(\varepsilon)$. Consequently $(a_k)_{k\in\mathbb{N}}$ converges to a in S and so is the space complete. $\square$

Exercise 1.2 Let $\Omega \subseteq \mathbb{R}^m$ be an open subset and let $(\Omega_n)_{n\in\mathbb{N}}$ be an exhaustion of Ω by open sets with compact closure, that is, each $\Omega_n \subseteq \mathbb{R}^m$ is open, $\overline{\Omega}_n$ is compact and contained in Ω, $\Omega_n \subseteq \Omega_{n+1}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$. Define

$$d(f,g) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{\|f-g\|_{C^0(\overline{\Omega}_n)}}{1+\|f-g\|_{C^0(\overline{\Omega}_n)}}$$

for every continuous, real-valued functions $f, g \in C^0(\Omega,\mathbb{R})$.

(a) Prove that d defines a metric in $C^0(\Omega,\mathbb{R})$.

(b) Prove that $(C^0(\Omega,\mathbb{R}), d)$ is a complete metric space.

(c) Let $C^0_c(\Omega,\mathbb{R})$ be the set of continuous functions with compact support in Ω. Prove that $C^0_c(\Omega,\mathbb{R})$ is dense in $(C^0(\Omega,\mathbb{R}), d)$.

Remark. The topology defined by d is called topology of the convergence on compact subsets of Ω. It is possible to prove that it does not depend on the chosen exhaustion.

Solution. Parts (a) and (b) of this exercise are solved similarly as in Exercises 1, so we point out the main differences.

(a) This is analogous to the solution in Exercise 1.

(b) *Claim: $(f_k)_{k\in\mathbb{N}} \subseteq C^0(\Omega,\mathbb{R})$ is a Cauchy sequence in $(C^0(\Omega,\mathbb{R}), d)$ if and only if, for every fixed k, $(f_k|_{\Omega_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0(\overline{\Omega}_n,\mathbb{R})$.*

The proof of this claim is similar to that of the claim in Exercise 1.

Note next that, since $C^0(\overline{\Omega}_n,\mathbb{R})$ is a Banach space, and, for every fixed n, $(f_k|_{\overline{\Omega}_n})_{k\in\mathbb{N}}$ is a Cauchy sequence in this space, it converges to some $\varphi_n \in C^0(\overline{\Omega}_n,\mathbb{R})$. Since for every k and n there holds $f_k|_{\overline{\Omega}_n} = f_k|_{\overline{\Omega}_{n+1}}$ in Ω_n, by the uniqueness of the limit and the fact that the Ω_n's are nested, it follows that $\varphi_n = \varphi_{n+1}$ in Ω_n, thus the definition

$$f(x) = \varphi_n(x) \quad \text{if } x \in \Omega_n,$$

is well-posed and f is a continuous function on Ω. Similarly as in Exercise 1, $(f_n)_{n\in\mathbb{N}}$ converges to f in $(C^0(\Omega,\mathbb{R}), d)$. So the space is complete.

(c) Let $f \in C^0(\Omega)$, and let $(\Omega_n)_{n\in\mathbb{N}}$ be an exhaustion of Ω as given by the exercise. We consider a sequence $(\eta_n)_{n\in\mathbb{N}}$ of smooth functions with compact support so that:

$$0 \le \eta_n \le 1, \qquad \eta_n \equiv 1 \text{ in } \overline{\Omega}_n, \qquad \operatorname{supp}(\eta_n) \subseteq \Omega_{n+1},$$

and we set

$$f_n(x) = f(x)\eta_n(x), \qquad \text{for } x \in \Omega.$$

Note that each f_n is continuous, has compact support and coincides with f in $\overline{\Omega}_n$. To see that such sequence converges to f in $(C^0(\Omega, \mathbb{R}), d)$, as in part (b) it is enough to prove that, for every fixed n, $(f_k|_{\overline{\Omega}_n})_{k\in\mathbb{N}}$ is Cauchy in $C^0(\overline{\Omega}_n, \mathbb{R})$. Since for $k \geq n$ $f_k|_{\overline{\Omega}_n}$ is identically equal to $f|_{\overline{\Omega}_n}$, this is trivially true. So f_n converges to f in $(C^0(\Omega, \mathbb{R}), d)$. □

Exercise 1.3 Let (X, d) be a metric space. Prove that the following are equivalent:

(a) The complement of every meager set is dense in X.

(b) The interior of every meager set is empty.

(c) The empty set is the only open and meager set.

(d) Countable intersections of open dense sets are dense.

Hint: recall that A is dense in X if and only if its complement has empty interior.

Remark. Thanks to Baire's Category Theorem, each of the above conditions are satisfied in a complete metric space.

Solution. We have:

- $(a) \Rightarrow (b)$: Assume that A is a meager set with nonempty interior. Then there exists $x \in X$ and $\varepsilon > 0$ so that $B_\varepsilon(x) \subseteq A$ and thus $\overline{A^c} \subseteq \overline{B_\varepsilon(x)^c} \neq X$. Thus A^c cannot be dense in X.

- $(b) \Rightarrow (c)$: If U is an open, meager set, then $U = \operatorname{int}(U) = \emptyset$.

- $(c) \Rightarrow (d)$: Let $A = \bigcap_{n\in\mathbb{N}} A_n$ be a countable intersection of open and dense sets. Consider its complement $A^c = \bigcup_{n\in\mathbb{N}} A_n^c$: each A_n^c is closed and has empty interior, so it is nowhere dense and A^c is meager. Consequently, $(A^c)^\circ$ is empty and thus A is dense.

- $(d) \Rightarrow (a)$: Let $A = \bigcup_{n\in\mathbb{N}} A_n$ be a meager set. Obviously $A \subseteq \bigcup_{n\in\mathbb{N}} \overline{A_n}$, and each $\overline{A_n}$ has emtpy interior. Thus $\bigcap_{n\in\mathbb{N}} (\overline{A_n})^c \subseteq A^c$, and since every $(\overline{A_n})^c$ is open and dense, so is their intersection, and so is A^c. □

Exercise 1.4 Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence. Define, for every $p \in [1,\infty]$,

$$\|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} = \begin{cases} \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup_{n\in\mathbb{N}} |x_n| & \text{if } p = \infty, \end{cases}$$

and let $\ell^p = \{(x_n)_{n\in\mathbb{N}} \mid \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} < \infty\}$. For every $p \in [1,\infty]$, $(\ell^p, \|\cdot\|_{\ell^p})$ is a Banach space.

Let now $1 \leq p < q \leq \infty$. Prove that:

(a) $\ell^p \subsetneq \ell^q$ and $\|(x_n)_{n\in\mathbb{N}}\|_{\ell^q} \leq \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p}$ for every $(x_n)_{n\in\mathbb{N}} \in \ell^p$.

(b) ℓ^p is meager in ℓ^q.

(c) $\bigcup_{1\leq p<q} \ell^p \subsetneq \ell^q$.

Hint for (b): The set $A_n = \{(x_n)_{n\in\mathbb{N}} \in \ell^q \mid \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} \leq n\} \subseteq \ell^q$ is closed in ℓ^q and has empty interior in ℓ^q.

Remark. Notice that $\ell^p = L^p(\mathbb{N}, \mathcal{A}, \mu)$, where $\mathcal{A}$ is the σ-algebra of all subsets of $\mathbb{N}$ and μ is the counting measure, i. e. $\mu(M)$ is the cardinality of M (possibly ∞).

Solution. (a) It suffices to prove the inequality $\|x\|_{\ell^q} \leq \|x\|_{\ell^p}$ for all $x \in \ell^p$. Since

$$(n^{-\frac{1}{p}})_{n\in\mathbb{N}} \in \ell^q \setminus \ell^p,$$

the inclusion is strict. Moreover $\|x\|_{\ell^q} \leq \|x\|_{\ell^p}$ if and only if $\|\lambda x\|_{\ell^q} \leq \|\lambda x\|_{\ell^p}$ for every $\lambda > 0$, so it suffices to prove $\|x\|_{\ell^q} \leq 1$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$ with $\|x\|_{\ell^p} = 1$.

Case $q = \infty$. For all $n \in \mathbb{N}$ we have

$$|x_n| = \left(|x_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} = \|x\|_{\ell^p} = 1.$$

Therefore, $\|x\|_{\ell^\infty} = \sup_{n\in\mathbb{N}} |x_n| \leq 1$.

Case $q < \infty$. The assumption $\|x\|_{\ell^p} = 1$ implies $|x_n| \leq 1$ for all $n \in \mathbb{N}$. Since $1 \leq p < q$, we have $|x_n|^q \leq |x_n|^p$ for all $n \in \mathbb{N}$. This implies the inequality

$$\|x\|_{\ell^q} = \left(\sum_{n\in\mathbb{N}} |x_n|^q\right)^{\frac{1}{q}} \leq \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{\frac{1}{q}} = \left(\|x\|_{\ell^p}^p\right)^{\frac{1}{q}} = 1^{\frac{p}{q}} = 1.$$

(b) In order to show that $A_n = \{x \in \ell^q \mid \|x\|_{\ell^p} \leq n\}$ is closed in $(\ell^q, \|\cdot\|_{\ell^q})$, we will prove that the limit of every ℓ^q-convergent sequence with elements in A_n is also in A_n.

Let $(a^{(k)})_{k\in\mathbb{N}}$ be a sequence of elements $a^{(k)} = (a_j^{(k)})_{j\in\mathbb{N}} \in A_n$. Suppose $b = (b_j)_{j\in\mathbb{N}} \in \ell^q$ satisfies $\lim\limits_{k\to\infty}\|a^{(k)} - b\|_{\ell^q} = 0$. Then, for every $j \in \mathbb{N}$,

$$|a_j^{(k)} - b_j| \leq \Big(\sum_{i\in\mathbb{N}} |a_i^{(k)} - b_i|^q\Big)^{\frac{1}{q}} = \|a^{(k)} - b\|_{\ell^q} \xrightarrow{k\to\infty} 0.$$

Let $N \in \mathbb{N}$ be arbitrary. By continuity of $|\cdot|^p \colon \mathbb{R} \to \mathbb{R}$, we have

$$\sum_{j=1}^{N} |b_j|^p = \lim_{k\to\infty} \sum_{j=1}^{N} |a_j^{(k)}|^p \leq \limsup_{k\to\infty} \|a^{(k)}\|_{\ell^p}^p \leq n^p$$

since the number of summands is finite. In the limit $N \to \infty$, we see $\|b\|_{\ell^p}^p \leq n^p$, which implies $b \in A_n$. Therefore, A_n is closed in $(\ell^q, \|\cdot\|_{\ell^q})$.

Assume by contradiction that A_n has non-empty interior in the ℓ^q- topology. Then there exist $a = (a_m)_{m\in\mathbb{N}} \in A_n$ and $\varepsilon > 0$ such that

$$B := \{x \in \ell^q \mid \|a - x\|_{\ell^q} < \varepsilon\} \subset A_n.$$

Consider $b = (b_m)_{m\in\mathbb{N}} \in \ell^q$ given by $b_m = m^{-\frac{1}{p}}$. Indeed, $\sum_{m=1}^{\infty} m^{-\frac{q}{p}} < \infty$ since $p < q$. We define $z = (z_m)_{m\in\mathbb{N}}$ by

$$z_m = a_m + \frac{\varepsilon b_m}{2\|b\|_{\ell^q}}.$$

Then $\|a - z\|_{\ell^q} = \frac{\varepsilon}{2}$ and $z \in B$. However, $b \notin \ell^p$ and $a \in \ell^p$ imply $z \notin \ell^p \supset A_n$ which contradicts $B \subset A_n$. Therefore, A_n has empty interior in $(\ell^q, \|\cdot\|_{\ell^q})$.

Being closed with empty interior, A_n is nowhere dense in $(\ell^q, \|\cdot\|_{\ell^q})$. Since $\ell^p = \bigcup_{n\in\mathbb{N}} A_n$ we may conclude that ℓ^p is meager in ℓ^q.

(c) Since $\ell^{p_1} \subset \ell^{p_2}$ for $p_1 < p_2$ by (a) we have

$$\bigcup_{p\in[1,q[} \ell^p = \bigcup_{p\in[1,q[\cap\mathbb{Q}} \ell^p.$$

By (b), the right hand side is a countable union of meager subsets of $(\ell^q, \|\cdot\|_{\ell^q})$ and therefore meager itself. Being complete, ℓ^q is not meager in $(\ell^q, \|\cdot\|_{\ell^q})$. Therefore, we may conclude strict inclusion

$$\bigcup_{p\in[1,q[\cap\mathbb{Q}} \ell^p \subsetneq \ell^q.$$

$\square$