Exercise 2.1 Let X be a vector space. An *algebraic basis* for X is a subset $E \subset X$ such that every $x \in X$ is uniquely given as *finite* linear combination of elements in E.

(a) Let $(X, \|\cdot\|)$ be a Banach space. Show that any algebraic basis for X is either finite or uncountable.

(b) Find an example of a normed space with a countably infinite algebraic basis.

Hint for (a): Assume that X has a countably infinite algebraic basis $\{e_1, e_2, \ldots\}$ and deduce a contradiction to Baire's Lemma by considering the sets $A_n = \operatorname{span}\{e_1, \ldots, e_n\}$.

Solution. (a) Assume by contradiction that X has a countably infinite algebraic basis $\{e_1, e_2, \ldots\}$. For $n \in \mathbb{N}$ we define the linear subspaces $A_n = \operatorname{span}\{e_1, \ldots, e_n\} \subset X$.

As finite dimensional subspace, A_n is closed. Suppose that A_n has non-empty interior. Then there exist $x \in A_n$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_n$. Since A_n is a linear subspace, we may subtract $x \in A_n$ from the elements in $B_\varepsilon(x)$ to obtain $B_\varepsilon(0) \subset A_n$. For the same reason,

$$A_n \supset \{\lambda y \mid \lambda > 0,\ y \in B_\varepsilon(x)\} = X.$$

This implies $\dim X \leq n$ which contradicts our assumption that the algebraic basis of X is infinite. Thus A_n must have empty interior and thus, being also closed, is nowhere dense. By assumption,

$$X = \bigcup_{n \in \mathbb{N}} A_n,$$

which implies that X is meager. Since X is complete, this contradicts Baire's Lemma.

(b) Let X be the space of polynomials $p\colon [0,1] \to \mathbb{R}$ with real coefficients endowed with the norm $\|\cdot\|_{C^0([0,1])}$. Let $f_n\colon [0,1] \to \mathbb{R}$ be given by the monomial $f_n(x) = x^n$. Then, $\{f_n \mid n \in \mathbb{N}\}$ is a countable algebraic basis for X.

Note that According to (a), $(X, \|\cdot\|_{C^0([0,1])})$ is necessarily incomplete. □

Exercise 2.2 Let $f \in C^0([0,\infty))$ be a continuous function satisfying

$$\forall t \in [0,\infty) : \lim_{n\to\infty} f(nt) = 0.$$

Prove that $\lim_{t\to\infty} f(t) = 0$.

Hint: Apply Baire's Lemma as in the proof of the uniform boundedness principle.

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Solution. Define $f_n(t) = |f(nt)|$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ and let

$$A_N := \bigcap_{n=N}^{\infty} \{t \in [0,\infty[\mid f_n(t) \leq \varepsilon\}.$$

Since f_n is continuous, the pre-image $f_n^{-1}([0,\varepsilon]) = \{t \in [0,\infty[\mid f_n(t) \leq \varepsilon\}$ is closed for all $n \in \mathbb{N}$. Thus, the set A_N is closed as intersection of closed sets. By assumption,

$$\forall t \in [0,\infty[\quad \exists N_t \in \mathbb{N} \quad \forall n \geq N_t : \quad f_n(t) \leq \varepsilon$$

which implies

$$[0,\infty[= \bigcup_{N=1}^{\infty} A_N.$$

Baire's Lemma applied to the complete metric space $([0,\infty[, |\cdot|)$ implies that there exists $N_0 \in \mathbb{N}$ such that A_{N_0} has non-empty interior, i. e. there exist $0 \leq a < b$ such that $]a,b[\subset A_{N_0}$. This implies

$$\begin{aligned} &\forall n \geq N_0 \quad \forall t \in]a,b[: \qquad f_n(t) \leq \varepsilon \\ \Leftrightarrow \quad &\forall n \geq N_0 \quad \forall t \in]na,nb[: \ |f(t)| \leq \varepsilon. \end{aligned}$$

If $n > \frac{a}{b-a}$, then $(n+1)a < nb$. For the intervals $J_{a,b}(n) :=]na,nb[$ this means that $J_{a,b}(n) \cap J_{a,b}(n+1) \neq \emptyset$. Let $N_1 > \max\{N_0, \frac{a}{b-a}\}$. Then, in particular,

$$\forall t > N_1 a : \qquad |f(t)| \leq \varepsilon.$$

This proves $\lim\limits_{t\to\infty} f(t) = 0$ since $\varepsilon > 0$ was arbitrary. $\square$

Exercise 2.3 Let

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N : \ x_n = 0\}$$

be the space of compactly supported sequences and

$$c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}.$$

be the space of sequences converging to zero.

(i) Show that $(c_c, \|\cdot\|_{\ell^\infty})$ is *not* complete. What is the completion of this space?

(ii) Prove the strict inclusion

$$\bigcup_{p\in[1,\infty)} \ell^p \subsetneq c_0.$$

Solution. (i) *First way:* If $e_n = (\delta_{k,n})_{k\in\mathbb{N}}$ denotes the sequence whose n-th term is 1 and all the other terms vanish, then $(e_n)_{n\in\mathbb{N}}$ is an algebraic and countable basis of c_0. By Exercise 1, c_0 cannot be complete.

Second way: Let $x_k = (x_{k,n})_{n\in\mathbb{N}} \in c_c$ be given by

$$x_{k,n} = \begin{cases} \frac{1}{n} & \text{for } n \le k, \\ 0 & \text{for } n > k. \end{cases}$$

Then $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $(c_c, \|\cdot\|_{\ell^\infty})$. However, its limit sequence x_∞ given by $x_{\infty,n} = \frac{1}{n}$ for all $n \in \mathbb{N}$ is not in c_c but in $c_0 \setminus c_c$, thus the space is not complete.

Note that, since ℓ^∞ is a complete space and $c_c \subset \ell^\infty$, the completion of c_c will still be a subspace of ℓ^∞.

Claim: c_0 is the completion of $(c_c, \|\cdot\|_{\ell^\infty})$.

Let us prove that $\overline{c_c} \subseteq c_0$. Let $x = (x_n)_{n\in\mathbb{N}} \in \overline{c_c}$. Then there exists a sequence of sequences $x_k = (x_{k,n})_{n\in\mathbb{N}} \in c_c$ such that $x_k \to x$ in ℓ^∞ as $k \to \infty$. Let $\varepsilon > 0$. In particular, there exists $K \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} |x_{K,n} - x_n| = \|x_K - x\|_{\ell^\infty} < \varepsilon$$

Since $x_K \in c_c$, there exists $N_0 \in \mathbb{N}$ such that $x_{K,n} = 0$ for all $n \ge N_0$. This implies that

$$\forall n \ge N_0 : \quad |x_n| \le \sup_{n\ge N_0} |0 - x_n| < \varepsilon.$$

We conclude that $x_n \to 0$ as $n \to \infty$ which means that $x \in c_0$.

Let us now prove that $c_0 \subseteq \overline{c_c}$. Let $x = (x_n)_{n\in\mathbb{N}} \in c_0$. Let $(x_k)_{k\in\mathbb{N}}$ be the sequence of sequences $x_k = (x_{k,n})_{n\in\mathbb{N}}$ in c_c given by

$$x_{k,n} = \begin{cases} x_n & \text{for } n \le k, \\ 0 & \text{for } n > k. \end{cases}$$

Let $\varepsilon > 0$. By assumption, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n| < \varepsilon$ for every $n \ge N_\varepsilon$.

$$\Rightarrow \ \forall k \ge N_\varepsilon : \quad \|x_k - x\|_{\ell^\infty} = \sup_{n>k} |0 - x_n| \le \varepsilon.$$

We conclude that $x_k \to x$ in ℓ^∞ as $k \to \infty$. Since $x \in c_0$ is arbitrary, $c_0 \subseteq \overline{c_c}$.

(ii) One inclusion is clear since any sequence $(x_n)_{n\in\mathbb{N}} \in \ell^p$ for $p \in [1,\infty)$, necessarily satisfies $x_n \to 0$ for $n \to \infty$ by standard facts concerning summable series.

To see that the inclusion is strict, consider the sequence $y = (y_n)_{n\in\mathbb{N}} \in c_0$ given by

$$y_n = \frac{1}{\log(n+1)}.$$

Then, $y \in c_0$ but $y \notin \ell^p$ for any $p \geq 1$: indeed, given any $p \geq 1$ there exists $N_p \in \mathbb{N}$ such that $\log(n+1) \leq n^{\frac{1}{p}}$ for every $n \geq N_p$ which allows the estimate

$$\sum_{n=1}^{\infty} \Big(\frac{1}{\log(n+1)}\Big)^p \geq \sum_{n=N_p}^{\infty} \Big(\frac{1}{n^{\frac{1}{p}}}\Big)^p = \sum_{n=N_p}^{\infty} \frac{1}{n} = \infty.$$

□

Exercise 2.4 Show that the subspaces

$$U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}:\ x_{2n} = 0\},$$
$$V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}:\ x_{2n-1} = n x_{2n}\}$$

are both closed in $(\ell^1, \|\cdot\|_{\ell^1})$ while the subspace $U \oplus V$ is not closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Hint. Prove that if any sequence $(x_k)_{k\in\mathbb{N}}$ of elements $x_k = (x^{k,n})_{n\in\mathbb{N}} \in \ell^1$ converges to $(x_n)_{n\in\mathbb{N}}$ in ℓ^1 for $k \to \infty$, then each entry $x_{n,k}$ converges in $\mathbb{R}$ to x_n for $k \to \infty$. For the second part, show first that c_c (see Exercise 2.1) is a subset of $U \oplus V$.

Solution.

Claim 1. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of sequences $x_k = (x_{k,n})_{n\in\mathbb{N}} \in \ell^1$ and let $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$. Then, the following implication is true.

$$\lim_{k\to\infty} \|x_k - x\|_{\ell^1} = 0 \quad \Rightarrow \ \forall n \in \mathbb{N}:\ \lim_{k\to\infty} |x_{k,n} - x_n| = 0.$$

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. By assumption, there exists $K_\varepsilon \in \mathbb{N}$ such that

$$\forall k \geq K_\varepsilon: \quad |x_{k,n} - x_n| \leq \sum_{n=1}^{\infty} |x_{k,n} - x_n| = \|x_k - x\|_{\ell^1} < \varepsilon.$$

□

Claim 2. $U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}:\ x_{2n} = 0\}$ is closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of sequences $x_k = (x_{k,n})_{n\in\mathbb{N}} \in U$ converging to $x = (x_n)_{n\in\mathbb{N}}$ in ℓ^1. By definition, $x_{2n,k} = 0$ for every $n \in \mathbb{N}$. According to Claim 1,

$$x_{2n} = \lim_{k\to\infty} x_{2n,k} = 0$$

for every $n \in \mathbb{N}$. Thus, $x \in U$. □

Claim 3. $V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}: \ x_{2n-1} = nx_{2n}\}$ is closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)} = (x_{k,n})_{n\in\mathbb{N}} \in V$ converging to $x = (x_n)_{n\in\mathbb{N}}$ in ℓ^1. By definition, $x_{k,2n-1} = nx_{k,2n}$ for every $n \in \mathbb{N}$. By Claim 1,

$$x_{2n-1} = \lim_{k\to\infty} x_{k,2n-1} = \lim_{k\to\infty} nx_{k,2n} = nx_{2n}$$

for every $n \in \mathbb{N}$. Thus, $x \in V$. □

Claim 4. $c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N: \ x_n = 0\} \subset U \oplus V$.

Proof. Let $x \in c_c$. Then, $x = u + v$ with $u = (u_m)_{m\in\mathbb{N}}$ and $v = (v_m)_{m\in\mathbb{N}}$ given by

$$u_m = \begin{cases} x_m - nx_{m+1}, & \text{if } m = 2n-1, \\ 0, & \text{if } m \text{ is even} \end{cases} \qquad v_m = \begin{cases} nx_{m+1}, & \text{if } m = 2n-1, \\ x_m, & \text{if } m \text{ is even.} \end{cases}$$

The assumption $x \in c_c$ implies $v, u \in c_c \subset \ell^1$. Then, $u \in U$ holds by construction and $v \in V$ follows from $v_{2n-1} = nx_{2n-1+1} = nx_{2n} = nv_{2n}$ for every $n \in \mathbb{N}$. □

Claim 5. The space c_c is dense in $(\ell^1, \|\cdot\|_{\ell^1})$.

Proof. Let $x \in \ell^1$. Let $x_k = (x_{k,n})_{n\in\mathbb{N}} \in c_c$ be given by

$$x_{k,n} = \begin{cases} x_n & \text{for } n < k, \\ 0 & \text{for } n \geq k. \end{cases}$$

Then,

$$\|x_k - x\|_{\ell^1} = \sum_{n=k}^{\infty} |x_n| \xrightarrow{k\to\infty} 0.$$ □

Claim 6. The sequence $x = (x_m)_{m\in\mathbb{N}}$ defined as follows is in ℓ^1 but not in $U \oplus V$.

$$x_m = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \frac{1}{n^2}, & \text{if } m = 2n. \end{cases}$$

Proof. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$ we have $x\in\ell^1$. Suppose $x=u+v$ for $u\in U$ and $v\in V$. Then, $u_{2n}=0$ implies $v_{2n}=x_{2n}=\frac{1}{n^2}$ for every $n\in\mathbb{N}$. By definition of V, we have $v_{2n-1}=nv_{2n}=\frac{1}{n}$ for every $n\in\mathbb{N}$. However, $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$ implies $v\notin\ell^1$ which contradicts the definition of V. □

Claims 4, 5 and 6 imply that

$$\overline{U\oplus V}\supset\overline{c_c}=\ell^1\supsetneq U\oplus V.$$

Therefore, $U\oplus V$ cannot be closed. □

Exercise 2.5 Let $(X,\|\cdot\|)$ be a normed vector space. Prove that the following statements are equivalent.

(i) $(X,\|\cdot\|)$ is a Banach space.

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\displaystyle\sum_{k=1}^{\infty}\|x_n\|<\infty$ the limit $\displaystyle\lim_{N\to\infty}\sum_{n=1}^{N}x_n$ exists.

Hint: A Cauchy sequence converges if and only if it has a convergent subsequence.

Solution. If $(X,\|\cdot\|)$ is a Banach space, and $(x_k)_{k\in\mathbb{N}}$ any sequence in X with $\sum_{k=1}^{\infty}\|x_k\|<\infty$, then $(s_n)_{n\in\mathbb{N}}$ given by $s_n=\sum_{k=1}^{n}x_k$ is a Cauchy sequence (and hence convergent) since by assumption, for every $\varepsilon>0$ there exists $N_\varepsilon\in\mathbb{N}$ such that for every $m\geq n\geq N_\varepsilon$,

$$\|s_m-s_n\|\leq\sum_{k=n+1}^{m}\|x_k\|\leq\sum_{k=N_\varepsilon+1}^{\infty}\|x_k\|<\varepsilon.$$

Conversely, we assume for every sequence $(x_k)_{k\in\mathbb{N}}$ in X that $\sum_{k=1}^{\infty}\|x_k\|<\infty$ implies convergence of $s_n=\sum_{k=1}^{n}x_k$ in X for $n\to\infty$. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy in X. Then,

$$\forall k\in\mathbb{N}\quad\exists N_k\in\mathbb{N}\quad\forall n,m\geq N_k:\quad\|y_n-y_m\|\leq 2^{-k}.$$

Without loss of generality, we can assume $N_{k+1}>N_k$. Let $x_k:=y_{N_{k+1}}-y_{N_k}$. Then,

$$\sum_{k=1}^{\infty}\|x_k\|=\sum_{k=1}^{\infty}\|y_{N_{k+1}}-y_{N_k}\|\leq\sum_{k=1}^{\infty}2^{-k}<\infty,$$

which by assumption implies that

$$s_n=\sum_{k=1}^{n}x_k=\sum_{k=1}^{n}(y_{N_{k+1}}-y_{N_k})=y_{N_{n+1}}-y_{N_1}$$

converges in X for $n\to\infty$. Hence, $(y_{N_n})_{n\in\mathbb{N}}$ is a convergent subsequence of $(y_n)_{n\in\mathbb{N}}$. Since $(y_n)_{n\in\mathbb{N}}$ is Cauchy, it converges to the same limit in X. Thus, X is complete. □