Exercise 6.1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let

$$A\colon D_A \subset X \to Y, \qquad B\colon D_B \subset X \to Y$$

be linear operators with $D_A \subset D_B$. Suppose that there exist $a \in (0,1)$ and $b \geq 0$ such that

$$\|Bx\|_Y \leq a\|Ax\|_Y + b\|x\|_X \quad \text{for every } x \in D_A.$$

Show that if A has closed graph, then $(A+B)\colon D_A \to Y$ has closed graph.

Solution. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D_A so that $x_n \to x$ in X and $(A+B)x_n \to y$ in Y. We are going to prove that $(A+B)x = y$.

Claim: $(A(x_n))_{n\in\mathbb{N}}$ is convergent in Y.

Proof. If suffices to prove that it is a Cauchy sequence. By triangle inequality and the assumption,

$$\begin{aligned}\|A(x_n - x_m)\|_Y &\leq \|(A+B)(x_n - x_m)\|_Y + \|B(x_n - x_m)\|_Y \\ &\leq \|(A+B)(x_n - x_m)\|_Y + a\|A(x_n - x_m)\|_Y + b\|x_n - x_m\|_X.\end{aligned}$$

thus

$$\|A(x_n - x_m)\|_Y \leq \frac{1}{1-a}\left(\|(A+B)(x_n - x_m)\|_Y + b\|x_n - x_m\|_X\right).$$

Since a, b are fixed and $(x_n)_{n\in\mathbb{N}}$, $((A+B)(x_n))_{n\in\mathbb{N}}$ are convergent (and thus Cauchy), this inequality implies that claim. □

Since the graph of A is closed by assumption, we have $x_n \to x$ in D_A and $Ax_n \to Ax$. Consequently, writing $Bx_n = (A+B)x_n - Ax_n$, we deduce that also $(B(x_n))_{n\in\mathbb{N}}$ is convergent and

$$\|B(x - x_n)\|_Y \leq a\|A(x - x_n)\| + b\|x - x_n\| \xrightarrow{n\to\infty} 0$$

which implies $Bx_n \to Bx$ in Y. Thus

$$y = \lim_{n\to\infty}(A+B)x_n = \lim_{n\to\infty} Ax_n + \lim_{n\to\infty} Bx_n = Ax + Bx = (A+B)x.$$

□

Exercise 6.2 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a closable linear operator. Assume that its closure $\overline{A}$ is injective. Show that then the inverse operator A^{-1} is closable and $\overline{A^{-1}} = (\overline{A})^{-1}$.

Solution. Since the closure $\overline{A}$ is assumed to be injective, A is injective and therefore has an inverse $A^{-1}\colon A(X) \to D_A$ defined over the image of A. The graph $\Gamma_{A^{-1}} \subseteq Y \times X$ of A^{-1} is given by

$$\Gamma_{A^{-1}} = \{(y,x) : y \in A(X),\ x = A^{-1}y\} = \{(Ax,x) : x \in D_A\}.$$

Now if a sequence $(y_n,x_n)_{n\in\mathbb{N}} \subseteq \Gamma_{A^{-1}}$ is so that $(y_n,x_n) \to (0,x)$, then $Ax_n = y_n \to 0$, hence $\overline{A}x = 0$. Since $\overline{A}$ is injective, $x = 0$ and so (Satz 3.4.1) A^{-1} is closable.

Similarly, for a sequence $(y_n,x_n)_{n\in\mathbb{N}} \subseteq \Gamma_{A^{-1}}$, there holds $(y_n,x_n) \to (y,x)$ if and only if $(x_n,Ax_n) \to (x,y)$, which is the same as as saying that $\overline{\Gamma_{A^{-1}}} = \Gamma_{\overline{A}^{-1}}$. $\square$

Exercise 6.3 Let $(X, \|\cdot\|_X)$ be a Banach space and $U \subset X$ a closed subspace. Recall the notion of topologically complemented space introduced in Exercise 4.3. Prove that:

(i) If $\dim(U) < \infty$, then U is topologically complemented.

(ii) If $\dim(X/U) < \infty$, then U is topologically complemented.

Solution. (i) It is sufficient to construct a projection map P as in Exercise 4.3. Let $e_1,\dots,e_n$ be a basis of the given finite-dimensional subspace $U \subset X$ let $f_1,\dots,f_n \in L(U,\mathbb{R})$ be the associated dual basis, uniquely defined by the conditions

$$f_i(e_j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

From the Hahn-Banach Theorem it follows that there exist extensions $F_i \in L(X;\mathbb{R})$ with $\|F_i\| = \|f_i\|$. We define

$$P\colon X \to X, \quad P(x) = \sum_{i=1}^n F_i(x)\, e_i.$$

Then P is linear and continuous, since

$$\|Px\|_X \le \left(\sum_{i=1}^n \|F_i\| \|e_i\|_X\right) \|x\|_X.$$

By construction, $P(X) \subset \operatorname{span}\{e_1,\dots,e_n\} = U$. By definition of f_i and F_i we have $P(e_i) = e_i$ for every $i \in \{1,\dots,n\}$. Therefore, $P(X) = U$. Finally, for every $x \in X$,

$$(P \circ P)(x) = P\Bigl(\sum_{i=1}^n F_i(x)\, e_i\Bigr) = \sum_{i=1}^n F_i(x)\, P(e_i) = \sum_{i=1}^n F_i(x)\, e_i = P(x).$$

It follows from Exercise 4.3 that U is topologically complemented.

(ii) Denote by $\pi\colon X \to X/U$, $\pi(x) = [x]$ the canonical quotient map. Since $\dim(X/U) = m < \infty$ we can choose a basis $[e_1], \dots, [e_m]$ for X/U and let as above $f_1, \dots f_m \in L(X/U, \mathbb{R})$ be the associated dual basis. Set $F_i := f_i \circ \pi \colon X \to \mathbb{R}$ and define

$$P\colon X \to X, \quad P(x) = \sum_{i=1}^{n} F_i(x)\, e_i.$$

Since $F_i(e_j) = f_i(\pi(e_j)) = f_i([e_j]) = \delta_{ij}$ we have $P \circ P = P$ as above. Since $[e_1], \dots, [e_m]$ is a basis for X/U, the representatives $e_1, \dots, e_m$ must be linearly independent in X. Therefore, $P(x) = 0$ implies $F_i(x) = f_i([x]) = 0$ for every $i \in \{1, \dots, n\}$ which in turn implies $[x] = [0]$ or $x \in U$. Conversely, $x \in U$ implies $\pi(x) = [0]$ and $P(x) = 0$. Thus we have shown $\ker(P) = U$. As in Exercise 4.3, we conclude that $(1 - P)$ is a continuous projection onto U which implies that U is topologically complemented. $\square$

Exercise 6.4 Let $(X, \|\cdot\|_X)$ be a normed space and let $f\colon X \to \mathbb{R}$ be linear and not identically zero.

(i) Show that f is *not* continuous if and only if $\ker(f)$ is dense in X.

(ii) Find an explicit example for X and f as in (i).

Solution. (i) *Necessity.* Suppose that f is not continuous. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X, which can be normed to $\|x_k\|_X = 1$ by linearity of f, such that $|f(x_k)| \to \infty$ as $k \to \infty$. Without loss of generality, we can assume $f(x_k) \neq 0$ for every $k \in \mathbb{N}$. Let now z be any element in X. For each $k \in \mathbb{N}$ we define

$$y_k := z - \frac{f(z)}{f(x_k)} x_k, \quad \text{so} \quad f(y_k) = f(z) - \frac{f(z)}{f(x_k)} f(x_k) = 0$$

and so $y_k \in \ker(f)$. We then see that

$$\|z - y_k\|_X = \left|\frac{f(z)}{f(x_k)}\right| \|x_k\|_X = \frac{|f(z)|}{|f(x_k)|} \xrightarrow{k\to\infty} 0$$

so $(y_k)_{k\in\mathbb{N}}$ converges to z and consequently $\ker(f)$ is dense in X.

Sufficiency. If f were continuous, than $\ker(f) = f^{-1}(\{0\})$ would be closed, and thus $\ker(f) = X$. But then f had to be identically 0, contradicting the hypothesis.

(ii) An example is given by $X = C^0([0,1],\mathbb{R})$ with the L^1-norm $\|\cdot\|_{L^1((0,1))}$ and $f = \delta_0$ the evaluation at 0:

$$f(\varphi) = \varphi(0), \quad \text{for every } \varphi \in C^0([0,1],\mathbb{R}).$$

Then $\ker f$ contains the space of all compactly supported functions $C_c^\infty((0,1),\mathbb{R})$, which is well-known to be dense in the L^1-topology. $\square$

Exercise 6.5 Let $(X, \|\cdot\|_X)$ be a normed space and let $\varphi\colon X \to \mathbb{R}$ be a continuous linear functional. Assume $N := \ker(\varphi) \subsetneq X$ and let $x_0 \in X \setminus N$. Prove that the following are equivalent.

(i) There exists $y_0 \in N$ with $\|x_0 - y_0\|_X = \operatorname{dist}(x_0, N)$.

(ii) There exists $x_1 \in X$ with $\|x_1\|_X = 1$ and $\|\varphi\| = |\varphi(x_1)|$.

Solution. Recall first that every element $x \in X$ is of the form $x = tx_0 + y$ with uniquely determined $t \in \mathbb{R}$ and $y \in N$.

"*(i)* $\Rightarrow$ *(ii)*". Call $d = \operatorname{dist}(x_0, N)$ and set $x_1 = \frac{1}{d}(x_0 - y_0)$. Then $\|x_1\|_X = 1$. Let $x \in X$ and let $t \in \mathbb{R}$ and $y \in N$ be as above. If $t = 0$, then $\varphi(x) = 0$ is not interesting. Therefore, we assume $t \neq 0$ and observe

$$\begin{aligned} |\varphi(x)| &= |\varphi(tx_0 + y)| = |\varphi(td\,x_1 + t\,y_0 + y)| = |t|d\,|\varphi(x_1)|, \\ \|x\|_X &= \|tx_0 + y\|_X = |t|\|x_0 + \tfrac{1}{t}y\|_X \geq |t| \inf_{\tilde{y} \in N} \|x_0 - \tilde{y}\|_X = |t|d. \end{aligned}$$

This implies that

$$\|\varphi\| \leq \sup_{x \in X} \frac{|\varphi(x)|}{\|x\|_X} \leq \frac{|t|d\,|\varphi(x_1)|}{|t|d} = |\varphi(x_1)|.$$

Since on the other hand $|\varphi(x_1)| \leq \|\varphi\|$, the inequalities above are in fact identities and we conclude $|\varphi(x_1)| = \|\varphi\|$.

"*(ii)* $\Rightarrow$ *(i)*" As above, we have $x_1 = tx_0 + y_1$ for uniquely determined $t \in \mathbb{R}$ and $y_1 \in N$. In fact, $t \neq 0$ since $|\varphi(x_1)| = \|\varphi\| \neq 0$. Therefore, $x_0 = \frac{1}{t}(x_1 - y_1)$. Let $y_0 := -\frac{1}{t}y_1 \in N$. Then

$$\|x_0 - y_0\|_X = \frac{1}{t} = \frac{\varphi(x_0)}{\varphi(x_1)},$$

and with $\|\varphi\| = \varphi(x_1)$ there results $\|\varphi\|\|x_0 - y_0\| = \varphi(x_0)$. But then for every $y \in N$ we have

$$\|\varphi\|\|x_0 - y_0\| = \varphi(x_0) = \varphi(x_0 - y) \leq \|\varphi\|\|x_0 - y\|,$$

and (i) follows. $\square$

Exercise 6.6 Let $(X, \|\cdot\|_X) = \Big(C^0([-1,1]), \|\cdot\|_{C^0([-1,1])}\Big)$. Recall the map $\varphi\colon X \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(t)\,\mathrm{d}t - \int_{-1}^0 f(t)\,\mathrm{d}t$$

from Exercise 3.2. Consider its kernel $N := \{f \in X \mid \varphi(f) = 0\}$ and some $x_0 \in X \setminus N$. Show that N is closed. Show that there does not exist any $y_0 \in N$ such that

$$\|x_0 - y_0\| = \operatorname{dist}(x_0, N).$$

Solution. From Exercise 3.2 (i) we know that $\varphi\colon X \to \mathbb{R}$ is a continuous linear functional. Therefore $N = \ker(\varphi)$ is a closed subspace of X. From Exercise 3.2 (ii) we know that $\|\varphi\| = 2$. From Exercise 3.2 (iii) we know that there does not exist any $x_1 \in X$ with $\|x_1\|_X = 1$ and $|\varphi(x_1)| = 2 = \|\varphi\|$. From Exercise 6.5 we know that this is equivalent to the statement that there does not exist any $y_0 \in N$ with $\|x_0 - y_0\| = \operatorname{dist}(x_0, N)$. $\square$

Hints to Exercises.

6.1 Start by proving that $(Ax_n)_{n\in\mathbb{N}}$ is Cauchy.

6.3 When working with finite dimensional vector spaces, you may always argue with bases and dual bases.

6.5 Start by recalling the first isomorphism theorem for linear maps.