Exercise 7.1 Let $(H, (\cdot,\cdot)_H))$ be a Hilbert space. Let $Y \subset H$ be any subspace and let $f\colon Y \to \mathbb{R}$ be a continuous linear functional. By the Hahn-Banach Theorem there exists an extension $F\colon H \to \mathbb{R}$ with $F|_Y = f$ and $\|F\| = \|f\|$. Prove that F is unique.

Solution. We first consider the case where Y is also closed subspace, so that $(Y, (\cdot,\cdot)_H)$ is itself a Hilbert space. Then by the Riesz Representation Theorem there exists a unique $h \in Y$ so that

$$f(y) = (y, h)_H \quad \text{for every } y \in Y.$$

Then f has an obvious linear and extension to H given by

$$F(x) = (x, h)_H \quad \text{for every } x \in H,$$

which satisfies $\|F\| = \|h\|_H = \|f\|$.

Claim 1: F is the unique extension of f to H with norm $\|f\|$.

Proof. If $\tilde{F}\colon H \to \mathbb{R}$ is another extension of f, by the Riesz Representation Theorem it must be of the form $\tilde{F}(x) = (x, \tilde{h})_H$ for some $\tilde{h} \in H$, and since $\|\tilde{F}\| = \|f\|$ there must hold $\|h\|_H = \|\tilde{h}\|_H$. Since $\tilde{F}|_Y = f = F|_Y$, we have

$$0 = F(y) - \tilde{F}(y) = (y, h - \tilde{h})_H, \quad \text{for every } y \in Y,$$

and so $h - \tilde{h} \in Y^\perp$. But then since $h \in Y$ there holds

$$\|h\|_H^2 = \|\tilde{h}\|_H^2 = \|\tilde{h} - h + h\|_H^2 = \|\tilde{h} - h\|_H^2 + \|h\|_H^2,$$

and so $\|\tilde{h} - h\|_H^2 = 0$ and this implies $\tilde{h} = h$. This proves Claim 1. □

Let us now consider the case where Y is not necessarily closed.

As a continuous linear operator, f is closable. Let $\overline{f}$ be its closure.

Claim 2: there holds $D(\overline{f}) = \overline{Y}$ and $\|\overline{f}\| = \|f\|$.

Proof. The inclusion $D(\overline{f}) \subseteq \overline{Y}$ is obvious; for the converse, consider $y \in \overline{Y}$ and a sequence $(y_k)_{k\in\mathbb{N}}$ in Y such that $y_k \to y$ as $k \to \infty$. From

$$|f(y_n) - f(y_m)| \le \|f\| \|y_n - y_m\|_H,$$

we conclude that $(f(y_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Thus, there exists $z \in \mathbb{R}$ such that $f(y_k) \to z$ as $k \to \infty$. This means that (y, z) is in the closure of the graph of f and we conclude $y \in D(\overline{f})$.

As for the equality of the norms, $\|\overline{f}\| \geq \|f\|$ is obvious; on the other hand if y and $(y_k)_{k\in\mathbb{N}}$ are as above,

$$|\overline{f}(y)| = \lim_{k\to\infty} |f(y_k)| \leq \lim_{k\to\infty} \|f\|\|y_k\|_H = \|f\|\|y\|_H,$$

which implies $\|\overline{f}\| \leq \|f\|$. This proves Claim 2. $\square$

Claim 2 implies that, in order to extend $f\colon Y \to \mathbb{R}$ to H, we can first uniquely extend to $\overline{f}\colon \overline{Y} \to \mathbb{R}$ without changing the norm and then extend to $F\colon H \to \mathbb{R}$. We may then reduce ourselves to the case where Y is a closed subspace, and so the conclusion is reached thanks to Claim 1. $\square$

Remark. It may be possible to reduce immediately to the case where Y is closed by observing that since f is linear and continuous then it is also uniformly continuous; consequently by a well-known extension theorem it can be extended to $\overline{Y}$, with the same norm. Such extension is unique and so coincides with $\overline{f}$.

Exercise 7.2 Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and $Q \subset H$ a nonempty convex subset. Let $x \in H$ with distance $d := \operatorname{dist}(x, Q)$ from Q. Prove that:

(i) Every sequence $(x_n)_{n\in\mathbb{N}}$ in Q with $\lim_{n\to\infty} \|x_n - x\|_H = d$ is a Cauchy sequence in H.

(ii) If Q is closed in H, then there exists a *unique* $y \in Q$ with $\|x - y\|_H = d$.

Solution. Without loss of generality, we can assume $x = 0$. Otherwise we apply the translation $y \mapsto y - x$ which is an isometry to the entire space H.

(i) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the convex set $Q \subset H$ with $\|x_n\| \to d = \operatorname{dist}(0, Q)$ as $n \to \infty$. By convexity of Q, we have that

$$x_n, x_m \in Q \quad \Rightarrow \quad \frac{x_n + x_m}{2} \in Q \quad \Rightarrow \quad \left\|\frac{x_n + x_m}{2}\right\|_H \geq \operatorname{dist}(0, Q) = d$$

for every $n, m \in \mathbb{N}$. The parallelogram identity (true in every Hilbert space):

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

then yields

$$\begin{aligned} \|x_n - x_m\|_H^2 &= 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - \|x_n + x_m\|_H^2 \\ &= 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - 4\left\|\frac{x_n + x_m}{2}\right\|_H^2 \leq 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - 4d^2. \end{aligned}$$

From $2\|x_n\|_H^2 \to 2d^2$ as $n \to \infty$, we conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in H.

(ii) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Q with $\|x_n\|_H \to d = \operatorname{dist}(0,Q)$ as $n \to \infty$. According to (i), it must be a Cauchy-sequence. Since H is complete and Q closed, $(x_n)_{n\in\mathbb{N}}$ converges to some $y \in Q$.

Suppose there is another point $\tilde{y} \in Q$ with $\|\tilde{y}\| = d$. Then, again by convexity and the parallelogram identity,

$$d^2 \le \left\|\frac{y+\tilde{y}}{2}\right\|_H^2 \le \left\|\frac{y+\tilde{y}}{2}\right\|_H^2 + \left\|\frac{y-\tilde{y}}{2}\right\|_H^2 = \frac{1}{2}\|y\|_H^2 + \frac{1}{2}\|\tilde{y}\|_H^2 = d^2$$

and we conclude that all the inequalities are in fact identities which implies

$$\left\|\frac{y-\tilde{y}}{2}\right\|_H^2 = 0.$$

Thus, $y = \tilde{y}$. □

Exercise 7.3 Let $(X, \|\cdot\|_X)$ be a normed space.

(i) Let A be a subset of X and let $\operatorname{conv}(A)$ denote its convex hull. Prove the following characterization:

$$\operatorname{conv}(A) = \left\{\sum_{k=1}^{n} \lambda_k x_k \;\middle|\; n \in \mathbb{N},\ x_1, \ldots, x_n \in A,\ \lambda_1, \ldots, \lambda_n \ge 0,\ \sum_{k=1}^{n} \lambda_k = 1\right\}.$$

(ii) Let $A, B \subset X$ be compact, convex subsets. Prove that $\operatorname{conv}(A \cup B)$ is compact.

Solution. (i) We denote by $\mathcal{C}$ the set on the right-side.

"($\subseteq$)": Since $A \subset \mathcal{C}$, it is enough to show that $\mathcal{C}$ is convex. In fact, given $0 < t < 1$ we have

$$t\sum_{k=1}^{n} \lambda_k x_k + (1-t)\sum_{k=1}^{m} \lambda'_k x'_k = \sum_{k=1}^{n+m} \mu_k y_k$$

with

$$0 \le \mu_k := \begin{cases} t\lambda_k & \text{if } k \in \{1, \ldots, n\}, \\ (1-t)\lambda'_{k-n} & \text{if } k \in \{n+1, \ldots, n+m\} \end{cases}$$

$$A \ni y_k := \begin{cases} x_k & \text{if } k \in \{1, \ldots, n\}, \\ x'_{k-n} & \text{if } k \in \{n+1, \ldots, n+m\} \end{cases}$$

and $\mu_1 + \ldots + \mu_{n+m} = t(\lambda_1 + \ldots + \lambda_n) + (1-t)(\lambda'_1 + \ldots + \lambda'_m) = t + (1-t) = 1.$

"($\supseteq$)": Let $x_1, \ldots, x_n \in A$ and let $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \ldots + \lambda_n = 1$. We can assume $\lambda_1 \neq 0$. Since $\operatorname{conv}(A)$ is convex and contains $x_1, x_2 \in A$, and since $\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} = 1$,

$$\operatorname{conv}(A) \ni \frac{\lambda_1}{\lambda_1+\lambda_2}x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}x_2 = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1+\lambda_2} =: y_2.$$

For the same reason,

$$\operatorname{conv}(A) \ni \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3}y_2 + \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}x_3 = \frac{\lambda_1 x_2 + \lambda_2 x_2 + \lambda_3 x_3}{\lambda_1+\lambda_2+\lambda_3} =: y_3.$$

Iterating this procedure, we obtain

$$\operatorname{conv}(A) \ni \frac{\lambda_1+\ldots+\lambda_{k-1}}{\lambda_1+\ldots+\lambda_k}y_{k-1} + \frac{\lambda_k}{\lambda_1+\ldots+\lambda_k}x_k = \frac{\lambda_1 x_1 + \ldots + \lambda_k x_k}{\lambda_1+\ldots+\lambda_k} =: y_k.$$

for every $k \in \{3, \ldots, n\}$. Since $\lambda_1 + \ldots + \lambda_n = 1$, we have $y_n = \lambda_1 x_1 + \ldots + \lambda_n x_n$.

(ii) First we prove the following characterization.

Claim: there holds

$$\operatorname{conv}(A \cup B) = \mathcal{D} := \bigcup_{(s,t)\in\triangle} (sA + tB),$$

where $\triangle := \{(s,t) \in \mathbb{R}^2 \mid s + t = 1,\ s,t \geq 0\}$.

Proof. "($\subseteq$)": By choosing $(s,t) = (1,0)$ we see $A \subset \mathcal{D}$. Analogously, $B \subset \mathcal{D}$, hence $A \cup B \subset \mathcal{D}$. If $x \in (\operatorname{conv}(A \cup B)) \setminus (A \cup B)$, then (i) implies that x is of the form

$$x = \sum_{k=1}^{j} s_k a_k + \sum_{k=j+1}^{n} t_k b_k,$$

where $j, n \in \mathbb{N}$, where $a_k \in A$ and $b_k \in B$ as well as $s_k, t_k \geq 0$ for every k and where $s_1 + \ldots + s_j + t_{j+1} + \ldots + t_n = 1$. Since $x \notin A \cup B$ by assumption, we have

$$s := \sum_{k=1}^{j} s_k > 0, \qquad\qquad t := \sum_{k=j+1}^{n} t_k > 0,$$

with $s+t=1$. Since A and B are both convex by assumption,

$$a := \frac{1}{s}\sum_{k=1}^{j} s_k a_k \in A, \qquad\qquad b := \frac{1}{t}\sum_{k=j+1}^{n} t_k b_k \in B,$$

and we have shown $x = sa + tb \in \mathcal{D}$.

"$\supseteq$" Let $a \in A$ and $b \in B$. Then $a, b \in \operatorname{conv}(A \cup B)$. Since $\operatorname{conv}(A \cup B)$ is convex, we must have $sa + tb \in \operatorname{conv}(A \cup B)$ for every $(s,t) \in \triangle$. This proves $\operatorname{conv}(A \cup B) \supseteq \mathcal{D}$. □

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{conv}(A\cup B)$. Then by the claim there exist $a_n \in A$ and $b_n \in B$ as well as $(s_n, t_n) \in \triangle$ such that $x_n = s_n a_n + t_n b_n$ for every $n \in \mathbb{N}$. Since $\triangle$ is compact in $\mathbb{R}^2$, a subsequence $((s_n, t_n))_{n\in\Lambda_1\subset\mathbb{N}}$ converges in $\triangle$; since A and B are compact there are subsequences $(a_n)_{n\in\Lambda_2}$ convergent in A and $(b_n)_{n\in\Lambda_3}$ convergent in B. Therefore, setting

$$\Lambda_4 = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3,$$

we obtain that $(x_n)_{n\in\Lambda_4}$ converges in $\mathcal{D}$, and so $\operatorname{conv}(A \cup B)$ is compact. □

Exercise 7.4 *(Lions-Stampacchia).* Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space Let $a\colon H\times H \to \mathbb{R}$ be a bilinear map so that:

(a) $a(x,y) = a(y,x)$ for every $x, y \in H$,

(b) there exists $\Lambda > 0$ so that $|a(x,y)| \le \Lambda \|x\|_H \|y\|_H$ for every $x, y \in H$,

(c) there exists $\lambda > 0$ so that $a(x,x) \ge \lambda \|x\|_H^2$ for every $x \in H$.

Let moreover $f\colon H \to \mathbb{R}$ be a continuous linear functional. Consider the map $J\colon H \to \mathbb{R}$ given by

$$J(x) = a(x,x) - 2f(x).$$

Prove that, for any $K \subset H$ be a nonempty closed, convex subset, there exists a *unique* $y_0 \in K$ such that *both* the following inequalities hold:

(i) $J(y_0) \le J(y)$ for every $y \in K$,

(ii) $a(y_0, y - y_0) \ge f(y - y_0)$ for every $y \in K$.

Solution. Note that since a is symmetric with

$$\lambda\|x\|_H^2 \leq a(x,x) \leq \Lambda\|x\|_H^2,$$

$a(\cdot,\cdot)$ induces on H a scalar product equivalent to $(\cdot,\cdot)_H$, whose associated norm we denote by $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$. Thus as a consequence of Riesz Representation Theorem (or by the Lax-Milgram Theorem) there exists a unique $x_0 \in H$ satisfying $f(x) = a(x_0, x)$ for all $x \in H$. In particular there holds

$$\begin{aligned} J(x) = a(x,x) - 2f(x) &= a(x,x) - 2a(x_0,x) \\ &= a(x - x_0, x) - a(x_0, x) \\ &= a(x - x_0, x - x_0) + a(x - x_0, x_0) - a(x, x_0) \\ &= a(x - x_0, x - x_0) - a(x_0, x_0). \end{aligned} \tag{1}$$

Since the topology induced by $\|\cdot\|_a$ is equivalent to the given one on H, K is also closed in $(H, \|\cdot\|_a)$ and we can apply Exercise 7.2 (b) in the Hilbert space $(H, a(\cdot,\cdot))$ and thus there exists a unique $y_0 \in K$ satisfying

$$\|x_0 - y_0\|_a = \inf_{y\in K}\|x_0 - y\|_a. \tag{2}$$

Now we turn to prove the required statements:

(i) By (1)-(2) we have that

$$J(y_0) = \|y_0 - x_0\|_a^2 - \|x_0\|_a^2 \leq \|y - x_0\|_a^2 - \|x_0\|_a^2 = J(y)$$

for every $y \in K$.

(ii) We need to show that

$$\begin{aligned} a(y_0, y - y_0) - f(y - y_0) &= a(y_0, y - y_0) - a(x_0, y - y_0) \\ &= a(y_0 - x_0, y - y_0) \end{aligned}$$

is non-negative for every $y \in K$. Since $y_0 \in K$ we have $ty + (1-t)y_0 \in K$ for every fixed $y \in K$ and every $t \in [0,1]$ by convexity of K. We consider the map $f\colon [0,1] \to \mathbb{R}$ given by

$$f(t) = \Big\|x_0 - \Big(ty + (1-t)y_0\Big)\Big\|_a^2 = \Big\|x_0 - y_0 + t(y_0 - y)\Big\|_a^2.$$

By (2), and since $ty + (1-t)y_0 \in K$ by convexity, f has a minimum at $t = 0$ which implies $f'(0) \geq 0$. We compute

$$f'(t) = 2a\Big(x_0 - y_0 + t(y_0 - y), y_0 - y\Big)$$

and so

$$f'(0) = 2a(x_0 - y_0, y_0 - y) = 2a(y_0 - x_0, y - y_0) \geq 0,$$

as desired. $\square$

Exercise 7.5 Consider the spaces

$$c_0 := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k = 0\Big\}, \qquad c := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k \text{ exists}\Big\}.$$

with norm $\|\cdot\|_{\ell^\infty}$.

(i) Is c a Banach space?

(ii) Show that the dual space of $(c_0, \|\cdot\|_{\ell^\infty})$ is *isometrically* isomorphic to $(\ell^1, \|\cdot\|_{\ell^1})$.

(iii) To which space is the dual space of $(c, \|\cdot\|_{\ell^\infty})$ isomorphic?

Solution. (i) Since ℓ^∞ is Banach, to show that c is complete it suffices to prove that it is closed in ℓ^∞. Let $\varkappa = (\varkappa_n)_{n\in\mathbb{N}} \in \bar{c}$. Then there exists a sequence of sequences $(x_k)_{k\in\mathbb{N}}$, $x_k = (x_{k,n})_{n\in\mathbb{N}} \in c$ such that

$$\sup_{n\in\mathbb{N}} |x_{k,n} - x_n| = \|x_k - \varkappa\|_{\ell^\infty} \xrightarrow{k\to\infty} 0.$$

Let $\varepsilon > 0$. Let $k_\varepsilon \in \mathbb{N}$ such that $\|x_{k_\varepsilon} - \varkappa\|_{\ell^\infty} < \varepsilon$. By definition, $x_{k_\varepsilon} \in c$ is a Cauchy-sequence. Let $N_\varepsilon \in \mathbb{N}$ such that $|x_{k_\varepsilon,n} - x_{k_\varepsilon,m}| < \varepsilon$ for every $m, n \geq N_\varepsilon$. Then

$$|x_n - x_m| \leq |x_n - x_{k_\varepsilon,n}| + |x_{k_\varepsilon,n} - x_{k_\varepsilon,m}| + |x_{k_\varepsilon,m} - x_m| < 3\varepsilon,$$

for every $m, n \geq N_\varepsilon$ which proves that x is a Cauchy-sequence. Therefore, $\varkappa \in c$.

(ii) The linear map $\Psi\colon \ell^1 \to c_0^*$ given by

$$\Psi(y)(x) = \sum_{n\in\mathbb{N}} x_n y_n,$$

where $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ is linear and well-defined, since we can estimate

$$|\Psi(y)(x)| \leq \sum_{n\in\mathbb{N}} |x_n y_n| \leq \|x\|_{\ell^\infty} \|y\|_{\ell^1},$$

and consequently also $\|\Psi(y)\|_{c_0^*} \leq \|y\|_{\ell^1}$. Let us show that in fact $\|\Psi(y)\|_{c_0^*} = \|y\|_{\ell^1}$ for every $y \in \ell^1$: given $y \in \ell^1$ we can apply $\Psi(y)$ to the sequence $x_k = (x_{k,n})_{n\in\mathbb{N}} \in c_0$ given by

$$x_{k,n} = \begin{cases} \frac{y_n}{|y_n|} & \text{if } n \leq k \text{ and } y_n \neq 0, \\ 0 & \text{else,} \end{cases}$$

which satisfies $\|x^{(k)}\|_{\ell^\infty} = 1$ and

$$\lim_{k\to\infty}|\Psi(y)(x_k)| = \lim_{k\to\infty}\sum_{n=1}^{k}|y_n| = \|y\|_{\ell^1} \quad \Rightarrow \|\Psi(y)\|_{c_0^*} = \sup_{\substack{x\in c_0\\ \|x\|_{\ell^\infty}=1}} |\Psi(y)(x)| \geq \|y\|_{\ell^1}.$$

Therefore, Ψ is an isometry and in particular is injective.

To prove that Ψ is surjective, we show first that every $f \in c_0^*$ is determined by its values on the elements $e_k = (e_{k,n})_{n\in\mathbb{N}} \in c_0$, where $e_k = (0,\ldots,0,1,0,\ldots)$ has the 1 at k-th position: in fact Given $x = (x_n)_{n\in\mathbb{N}} \in c_0$, we have

$$\Big\|x - \sum_{k=1}^{N} x_k e_k\Big\|_{\ell^\infty} = \sup_{n>N}|x_n| \xrightarrow{N\to\infty} 0.$$

and so continuity and linearity of f implies

$$f(x) = \lim_{N\to\infty} f\Big(\sum_{k=1}^{N} x_k e_k\Big) = \lim_{N\to\infty}\sum_{k=1}^{N} x_k f(e_k).$$

Given $f \in c_0^*$ we claim that $y := (f(e_k))_{k\in\mathbb{N}} \in \ell^1$ and $\Psi(y) = f$. Indeed, for any $N \in \mathbb{N}$

$$\sum_{k=1}^{N}\Big|f(e_k)\Big| = \sum_{k=1}^{\infty} x_{N,k} f(e_k) = f(x_N) \leq \|f\|_{c_0^*},$$

where $x_N = (x_{N,k})_{k\in\mathbb{N}} \in c_0$ with $\|x_N\|_{\ell^\infty} \leq 1$ is defined by

$$x_{N,k} = \begin{cases} \frac{f(e_k)}{|f(e_k)|} & \text{if } k \leq N \text{ and } f(e_k) \neq 0,\\ 0 & \text{else.}\end{cases}$$

Since N is arbitrary, we conclude $y \in \ell^1$. Moreover, given any $x = (x_k)_{k\in\mathbb{N}} \in c_0$ and y as above, we have

$$\Psi(y)(x) = \sum_{k\in\mathbb{N}} x_k y_k = \sum_{k\in\mathbb{N}} x_k f(e_k) = f(x)$$

which shows that Ψ is surjective.

(iii) The dual space of $(c, \|\cdot\|_{\ell^\infty})$ is also isomorphic to $c_0^* \cong \ell^1$ but not isometrically. To construct an isomorphism $\Phi\colon c^* \to c_0^*$, we first consider the linear map $T\colon c \to c_0$ given by

$$Tx = \Big(\lim_{n\to\infty} x_n, (x_1 - \lim_{n\to\infty} x_n), (x_2 - \lim_{n\to\infty} x_n), \ldots\Big).$$

By definition of c and c_0, the map T is well-defined. T is continuous since

$$|\lim_{n\to\infty} x_n| \leq \|x\|_{\ell^\infty} \;\Rightarrow\; \|Tx\|_{\ell^\infty} \leq 2\|x\|_{\ell^\infty}.$$

Moreover, T is invertible with inverse $S\colon c_0 \to c$ given by

$$S(y) \mapsto \Big((y_2+y_1),(y_3+y_1),(y_4+y_1),\ldots\Big).$$

Indeed, $STx = x$ is immediate and $TSy = y$ follows from $\lim_{n\to\infty}(y_n + y_1) = y_1$. Since $\|Sy\|_{\ell^\infty} \leq 2\|y\|_{\ell^\infty}$, the map S is also continuous. So we define $\Phi\colon c^* \to c_0^*$ by

$$\Phi(f) = f \circ S.$$

As composition of linear maps, Φ is linear. It is also continuous since

$$|(\Phi f)(y)| = |f(Sy)| \leq \|f\|_{c^*}\|Sy\|_{\ell^\infty} \leq 2\|f\|_{c^*}\|y\|_{\ell^\infty}$$

By the construction above, Φ bijective with continuous inverse $\Phi^{-1}(g) = g \circ T$.

$\square$

Hints to Exercises.

7.1 Start by considering the case where Y is a *closed* subspace, and use Riesz representation theorem. Then reduce the general case to this one by considering the closure of f.

7.2 Use the parallelogram identity.

7.3 For (ii), show that

$$\operatorname{conv}(A \cup B) = \bigcup_{\substack{s,t \geq 0 \\ s+t=1}} (sA + tB),$$

and then prove that the right-hand side is compact.

7.4 Note first that $a(\cdot,\cdot)$ is a scalar product topologically equivalent to $(\cdot,\cdot)_H$, then use Riesz Representation Theorem for f with this scalar product and Exercise 7.2.

7.5 For (ii), compare with Satz 4.4.1 (dual of L^p). For (iii): how can one transform a sequence in c into a sequence in c_0?