Exercise 8.1 Let $(X, \|\cdot\|_X)$ be a normed space and let $\emptyset \neq Q \subset X$ be an open, convex subset containing the origin. Prove that Q can be written as intersection of open affine half-spaces, namely that there exists a subset $\Upsilon \subset X^*$ such that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\}.$$

Solution. We show that the statement holds with

$$\Upsilon = \{f \in X^* \mid \forall x \in X: \; f(x) \leq p(x)\},$$

where $p : X \to \mathbb{R}$ is the Minkowski functional for Q.

Note first of all that since Q is open and contains the origin, there exists $r > 0$ such that $B_r(0) \subset Q$. Thus, $\frac{1}{\lambda}x \in Q$ with $\lambda = \frac{2}{r}\|x\|_X$ and

$$p(x) \leq \tfrac{2}{r}\|x\|_X \quad \text{for every } x \in X.$$

"$(\subseteq)$": Let $x \in Q$. Since Q is open, we have $p(x) < 1$. For every $f \in \Upsilon$ we have $f(x) \leq p(x)$ by definition. This proves $f(x) < 1$ for every $f \in \Upsilon$.

"$(\supseteq)$" It is equivalent to prove that for every $x_0 \notin Q$ there exists $f \in \Upsilon$ with $f(x_0) \geq 1$. To this aim we define the functional

$$\ell\colon \operatorname{span}(\{x_0\}) \to \mathbb{R}, \quad \ell(tx_0) = t.$$

Since Q is convex and contains the origin, we have $p(x_0) \geq 1$. In particular, we have

$$\begin{aligned} \forall t \geq 0: \quad & \ell(tx_0) = t \leq t\,p(x_0) = p(tx_0), \\ \forall t < 0: \quad & \ell(tx_0) = t < 0 \leq p(tx_0). \end{aligned}$$

The Hahn-Banach theorem implies that there exists a linear continuous functional $f\colon X \to \mathbb{R}$ which agrees with ℓ on $\operatorname{span}(\{x_0\})$ with $f(x) \leq p(x) \leq \frac{2}{r}\|x\|_X$, and therefore $f \in \Upsilon$. Since $f(x_0) = 1$, the claim follows. □

Exercise 8.2 Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $\emptyset \neq K \subset H$ be a closed, convex subset. Denote by $P\colon H \to K$ be the projection operator onto K which maps $x \in H$ to the unique point $P(x) \in K$ with $\|x - Px\|_H = \operatorname{dist}(x, K)$ constructed in Exercise 7.7 (ii).

(i) For every $x_1, x_2 \in H$ prove the inequality

$$\|P(x_1) - P(x_2)\|_H \leq \|x_1 - x_2\|_H.$$

(ii) Prove that

$$K = \bigcap_{x \in H} \{y \in H \mid (x - P(x), P(x) - y)_H \geq 0\}.$$

Solution. (i) We first note that given any $x_0 \in H$ we have

$$(P(x_0) - x_0, y - P(x_0))_H \geq 0 \quad \text{for every } y \in K. \tag{1}$$

This inequality is proved in Exercise 7.4 (ii) for $a(\cdot,\cdot) = (\cdot,\cdot)_H$. Let $x_1, x_2 \in H$. Applying (1) twice we see that

$$\begin{aligned}
\|P(x_1) - P(x_2)\|_H^2 &= (P(x_1) - P(x_2), P(x_1) - P(x_2))_H \\
&= \quad (P(x_1) - x_1, P(x_1) - P(x_2))_H + (x_1, P(x_1) - P(x_2))_H \\
&\quad - (P(x_2) - x_2, P(x_1) - P(x_2))_H - (x_2, P(x_1) - P(x_2))_H \\
&\leq (x_1 - x_2, P(x_1) - P(x_2))_H,
\end{aligned}$$

thus with Caychy-Schwarz

$$\|Px_1 - Px_2\|_H^2 \leq \|x_1 - x_2\|_H \|Px_1 - Px_2\|_H,$$

which yields the desired inequality.

(ii) "($\subseteq$)": Let $y \in K$. Then $(Px - x, y - Px)_H \geq 0$ for any $x \in H$ by (1).

"($\supseteq$)": Let $y \in H \setminus K$. Then, choosing $x = y$, we have $Py \neq y$ which implies

$$(Py - y, y - Py)_H = -\|Py - y\|_H^2 < 0$$

and shows that y is not element of the set on the right hand side. $\square$

Exercise 8.3 A normed space $(X, \|\cdot\|_X)$ is called *strictly convex* if, for every $x, y \in X$ with $x \neq y$ and $\|x\|_X = 1 = \|y\|_X$, there holds

$$\|\lambda x + (1 - \lambda)y\|_X < 1 \quad \text{for every } \lambda \in (0, 1)$$

Let $(X, \|\cdot\|_X)$ be a normed space.

(i) Prove that if X^* is strictly convex, then for all $x \in X$ there exists a *unique* $x^* \in X^*$ with $\|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2$.

(ii) Find a counterexample to (i) when X^* is not strictly convex.

Solution. (i) The existence of x^* is given by Satz 4.2.1 and we need to show uniqueness. Given $0 \neq x \in X$, let $x^* \in X^*$ and $y^* \in X^*$ satisfy

$$\|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2 = y^*(x) = \|y^*\|_{X^*}^2.$$

Then

$$\|x\|_X^2 = \frac{1}{2}\Big(x^*(x) + y^*(x)\Big) = \left(\frac{1}{2}x^* + \frac{1}{2}y^*\right)(x) \leq \left\|\frac{1}{2}x^* + \frac{1}{2}y^*\right\|_{X^*} \|x\|_X,$$

hence

$$1 \leq \left\|\frac{x^*}{2\|x\|_X} + \frac{y^*}{2\|x\|_X}\right\|_{X^*}.$$

If $x^* \neq y^*$ were true, since

$$\left\|\frac{x^*}{\|x\|_X}\right\|_{X^*} = 1 = \left\|\frac{y^*}{\|x\|_X}\right\|_{X^*},$$

then with $\lambda = \frac{1}{2}$ in the definition of strict convexity we would obtain

$$\left\|\frac{x^*}{2\|x\|_X} + \frac{y^*}{2\|x\|_X}\right\|_{X^*} < 1,$$

thus contradicting the previous inequality.

(ii) Consider the space $(\mathbb{R}^2, \|\cdot\|_\infty)$, where $\|p\|_\infty = \max\{|p_1|, |p_2|\}$. Let $x = (1,1)$. Then, $\|x\|_\infty = 1$ and the functionals

$$x^* \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (p_1, p_2) \mapsto p_1 \qquad\qquad y^* \colon \mathbb{R}^2 \to \mathbb{R}^2 \quad (p_1, p_2) \mapsto p_2$$

both satisfy $x^*(x) = y^*(x) = 1 = \|x\|_\infty^2$ and

$$\|x^*\|_{X^*} = \sup_{\|p\|_\infty \leq 1} |x^*(p)| = \sup_{|p_1|,|p_2| \leq 1} |p_1| = 1 = \sup_{|p_1|,|p_2| \leq 1} |p_2| = \|y^*\|_{X^*}.$$

□

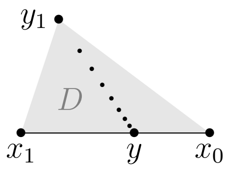
Exercise 8.4

(i) Let $K \subset \mathbb{R}^2$ be a closed, convex subset. Prove that the set E of all extremal points of K is closed.

(ii) Is the statement of (i) also true in $\mathbb{R}^3$?

Solution. (i) It is clear that the set E of extremal points of the closed, convex subset $K \subset \mathbb{R}^2$ must be a subset of the boundary ∂K of K because the center of every ball contained in K is a convex combination of other points in this ball.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in E which converges to some $y \in K$. Suppose $y \notin E$. Then there exist distinct points $x_1, x_0 \in K$ and some $0 < \lambda < 1$ such that $\lambda x_1 + (1-\lambda)x_0 = y$. For any $n \in \mathbb{N}$, the point y_n is extremal and therefore cannot lie on the segment between x_1 and x_0. Intuitively, the sequence $(y_n)_{n\in\mathbb{N}}$ must approach y from "above" or "below" this segment. By restriction to a subsequence, we can assume that all y_n strictly lie on the same side of the the affine line through x_1 and x_0. By convexity of K, the triangle $D = \operatorname{conv}\{x_1, x_0, y_1\}$ is a subset of K. The arguments above and convergence $y_n \to y$ imply that for $n \in \mathbb{N}$ sufficiently large, y_n is in the interior of D and thus in the interior of K. This however contradicts $y_n \in E \subset \partial K$. We conclude $y \in E$ which proves that E is closed.



(ii) The set of extremal points of a closed, convex subset in $\mathbb{R}^3$ is not necessarily closed: Let $S = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $p_\pm = (0, 1, \pm 1)$. The set of extremal points of $\operatorname{conv}(S \cup \{p_+, p_-\})$ is $E = \{p_+, p_-\} \cup S \setminus p_0$, where $p_0 = (0, 1, 0) = \frac{1}{2}p_+ + \frac{1}{2}p_-$. □

Exercise 8.5 Let X be vector space and let $K \subset X$ be a convex subset with more than one element.

(i) Given an extremal subset $M \subset K$ of K, prove that $K \setminus M$ is convex.

(ii) If $N \subset K$ and $K \setminus N$ are both convex, does it follow that N is extremal?

(iii) Prove that $y \in K$ is an extremal point of K if and only if $K \setminus \{y\}$ is convex.

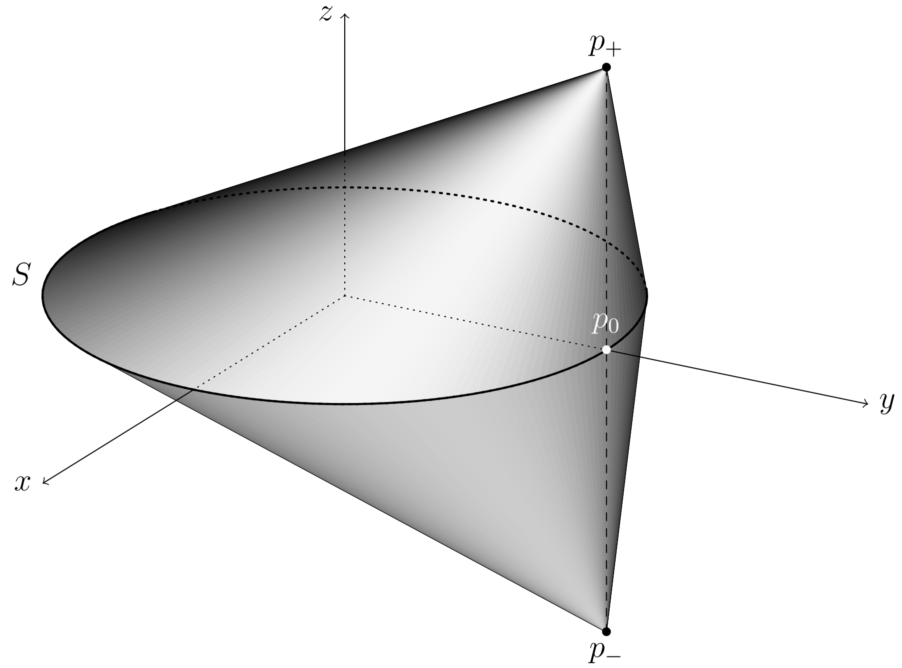
Solution. (i) Let $K \subset X$ be convex and $M \subset K$ an extremal subset of K. Suppose, $K \setminus M$ is not convex. Then there are points $x_1, x_0 \in K \setminus M$ such that $x := \lambda x_1 + (1-\lambda)x_0 \notin K \setminus M$ for some $0 < \lambda < 1$. Since K is convex,

$x \in K$ and hence $x \in M$. However, this contradicts $x_1, x_0 \notin M$ by definition of extremal subset.

(ii) No, the interval $K = [-1, 1] \subset \mathbb{R}$, the subset $N = [-1, 0] \subset K$ and the difference $K \setminus N =]0, 1]$ are all convex but N is not an extremal subset of K since $\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0 \in N$ but $1 \notin N$.

(iii) If $y \in K$ is an extremal point of K, then $\{y\} \subset K$ is an extremal subset of K and (i) implies that $K \setminus \{y\}$ is convex. Conversely, if $y \in K$ is not an extremal point of K, then by definition there exist $x_0 \in K \setminus \{y\}$, $x_1 \in K$ and some $0 < \lambda < 1$ such that $y = \lambda x_1 + (1 - \lambda)x_0$; since however $\{y\}$ consist of a single point it must also hold that $x_1 \in K \setminus \{y\}$, and thus $K \setminus \{y\}$ is not convex. □

Figure 1: Picture for Exercise 8.4 (ii)



Hints to Exercises.

8.1 Define Υ by means of the Minkowski functional for Q (used in the proof of Satz 4.5.1) $p\colon X \to \mathbb{R}$ given by

$$p(x) = \inf\{\lambda > 0 \mid \tfrac{1}{\lambda}x \in Q\}.$$

Note that p is sublinear and combine this with the Hahn-Banach theorem.

8.2 Use that for any $y \in K$ any $x_0 \in H$ there holds

$$(Px_0 - x_0, y - Px_0)_H \geq 0,$$

which is special case of the an inequality shown in Exercise 7.4 (ii) with $a(\cdot,\cdot) = (\cdot,\cdot)_H$.

8.4 For (i), show that E is a subset of the boundary ∂K. Argue by contradiction supposing E is not closed. For (ii), draw a picture.