Exercise 9.1 Let ℓ^∞ be the space of real-valued bounded sequences and let c be the subspace of converging sequences. Consider the functional

$$\lim : c \to \mathbb{R} \quad \lim(x_n) = \lim_{n\to\infty} x_n.$$

(i) Prove that it extends to a continuous linear functional $\mathfrak{lim} : \ell^\infty \to \mathbb{R}$ with norm $\|\mathfrak{lim}\| = 1$ and that there holds

$$\liminf_{n\to\infty} x_n \leq \mathfrak{lim}(x_n) \leq \limsup_{n\to\infty} x_n.$$

(ii) Use such construction to prove that the space ℓ^1 is not reflexive.

Solution. (i) Trivially in c there holds $\lim_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$ and since for every $\alpha \in \mathbb{R}$, and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell^\infty$ there holds

$$\limsup_{n\to\infty}(\alpha x_n) = \alpha \limsup_{n\to\infty} x_n,$$
$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,$$

we have that the functional $\limsup : \ell^\infty \to \mathbb{R}$ is sublinear. Consequently by the Hahn-Banach theorem we deduce the existence of a linear functional $\mathfrak{lim} : \ell^\infty \to \mathbb{R}$ with

$$\mathfrak{lim}(x_n) \leq \limsup_{n\to\infty} x_n,$$

for every $(x_n)_{n\in\mathbb{N}} \subset \ell^\infty$. To see that the opposite inequality with "lim inf" holds, it suffices note that, by linearity, there holds

$$\mathfrak{lim}(x_n) = -\mathfrak{lim}(-x_n) \geq -\limsup_{n\to\infty}(-x_n) = \liminf_{n\to\infty}(x_n).$$

To prove that $\mathfrak{lim}$ is continuous with norm 1, note that, on the one hand, $\lim : c \to \mathbb{R}$ has norm 1 and so $\|\mathfrak{lim}\| \geq 1$. On the other hand, if $(x_n)_{n\in\mathbb{N}} \subset \ell^\infty$ is any sequence with $\|(x_n)\|_{\ell^\infty} = \sup_{n\in\mathbb{N}} |x_n| = 1$, then

$$-1 \leq \liminf_{n\to\infty}(x_n) \leq \mathfrak{lim}(x_n) \leq \limsup_{n\to\infty}(x_n) \leq 1$$

from which $\|\mathfrak{lim}\| \leq 1$ follows.

(ii) It suffices to show that $\mathfrak{lim}$ does not corresponds to any sequence in ℓ^1 via the canonical injection $\ell^1 \hookrightarrow (\ell^1)^{**} = (\ell^\infty)^*$.

Suppose by contradiction that $(\mathfrak{x}_n)_{n\in\mathbb{N}}$ is the corresponding sequence. Since $\mathfrak{lim} \not\equiv 0$ such sequence must to be different from the zero sequence; At the same time however $\mathfrak{lim}$ vanishes when tested against sequences in ℓ^1 since they are bounded and convergent with zero limit. Thus testing $\mathfrak{lim}$ against $(\mathfrak{x}_n)_{n\in\mathbb{N}}$ (recall that $\ell^1 \subset \ell^\infty$) would give

$$0 = \mathfrak{lim}(\mathfrak{x}_n) = \sum_{n=1}^{\infty} |\mathfrak{x}_n|^2,$$

and so $(\mathfrak{x}_n)_{n\in\mathbb{N}}$ must be the zero sequence. The contradiction is reached. $\square$

Exercise 9.2 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $T\colon X \to Y$ be a linear operator. Prove that the following statements are equivalent.

(i) T is continuous.

(ii) T is weak-weak sequentially continuous, namely if $(x_n)_{n\in\mathbb{N}}$ is any weakly converging sequence X, then Tx_n is weakly convergent in Y.

Solution. "(i) $\Rightarrow$ (ii)" Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \xrightarrow{\mathrm{w}} x$ for some $x \in X$. Let $f \in Y^*$ be arbitrary. If $T\colon X \to Y$ is a continuous linear operator, then $f \circ T \in X^*$ and weak convergence of $(x_n)_{n\in\mathbb{N}}$ implies

$$\lim_{n\to\infty} f(Tx_n) = \lim_{n\to\infty} (f \circ T)(x_n) = (f \circ T)(x) = f(Tx)$$

which proves weak convergence of $(Tx_n)_{n\in\mathbb{N}}$ in Y.

"(ii) $\Rightarrow$ (i)" If the linear operator $T\colon X \to Y$ is not continuous, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\|x_n\|_X \leq 1$ and $\|Tx_n\|_Y \geq n^2$ for every $n \in \mathbb{N}$. Then $\frac{1}{n}x_n \to 0$ in X (in particular weakly) but $(T(\frac{1}{n}x_n))_{n\in\mathbb{N}}$ is unbounded in Y and therefore cannot be weakly convergent (Satz 4.6.1.). $\square$

Exercise 9.3 Let $(X, \|\cdot\|_X)$ be a finite-dimensional normed space. Prove that then strong and weak topologies coincide, namely that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is weakly convergent if and only if it is strongly convergent.

Solution. Clearly we only need to prove that weak convergence implies strong convergence. Let $e_1, \ldots, e_d$ be a basis for X. and let $e_1^*, \ldots, e_d^*$ be the corresponding dual basis. Since every norm in a finite-dimensional space is equivalent to the Euclidean norm, we may as well suppose, having written $x = \sum_{i=1}^d x^i e_i$ that

$$\|x\|_X = \left(\sum_{i=1}^d |x^i|^2 \right)^{1/2}.$$

Note that $e_k^* \in X^*$ since for every $x \in X$ we may write $x = \sum_{i=1}^d x^i e_i$ for some $x^k \in \mathbb{R}$ and so $|e_k^*(x)| = |x^i| \le \|x\|_X$.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \xrightarrow{w} x$ for some $x \in X$ as $n \to \infty$, then

$$\forall k \in \{1, \ldots, d\}: \quad \lim_{n\to\infty} x_n^k = \lim_{n\to\infty} e_k^*(x_n) = e_k^*(x) = x^k.$$

This implies $\|x_n - x\|_X \to 0$. □

Exercise 9.4 Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and let $(e_n)_{n\in\mathbb{N}} \subseteq X$ be an *orthonormal system for* H, that is, a countable set of elements so that

$$\langle e_j, e_k\rangle = \delta_{jk} \quad \text{for every } j, k \in \mathbb{N}.$$

(i) Prove $e_n \xrightarrow{w} 0$ as $n \to \infty$.

(ii) Suppose now that $(e_n)_{n\in\mathbb{N}}$ forms a *Hilbert basis for* H, i.e. that $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is a dense subspace of H. Prove that for every $x \in H$ there holds

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n, \tag{1}$$

and that *Parseval's Identity* holds:

$$\|x\|_H = \left(\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2\right)^{1/2}. \tag{2}$$

Solution. We prove first Bessel's inequality. Let $x \in H$ and $N \in \mathbb{N}$. Define

$$x_N := x - \sum_{n=1}^{N} \langle e_n, x\rangle e_n.$$

Then there holds $x_N \perp e_j$ for every $j \in \{1, 2, \ldots, N\}$ and so by using the orthonormality relations we deduce

$$\begin{aligned}
\sum_{n=1}^{N} |\langle x, e_n\rangle|^2 &\le \|x_N\|_H^2 + \sum_{n=1}^{N} |\langle x, e_n\rangle|^2 \\
&= \|x_N\|_H^2 + \left\|\sum_{n=1}^{N} \langle x, e_n\rangle e_n\right\|_H^2 \\
&= \left\|x_N + \sum_{n=1}^{N} \langle x, e_n\rangle e_n\right\|_H^2 = \|x\|_H^2.
\end{aligned}$$

(i) By Bessel's inequality

$$\sum_{n=0}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|_H^2,$$

thus it must hold $\langle x, e_n \rangle \to 0$ as $n \to \infty$ for any $x \in H$.

(ii) Define $s_N = \sum_{n=1}^{N} \langle x, e_n \rangle e_n$. By Bessel's inequality the sequence $(s_N)_{N \in \mathbb{N}}$ is Cauchy and thus convergent in H. By orthonormality we have

$$\langle x - s_N, e_n \rangle = 0 \quad \text{for } n \leq N,$$

and thus, passing to the limit in N and by linearity, it follows that

$$\langle x - s, v \rangle = 0 \quad \text{for every } v \in \operatorname{span}\{e_n : n \in \mathbb{N}\}.$$

but since $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is dense in X, it follows that $\langle x - s, y \rangle = 0$ for every $y \in X$, and thus that (1) holds. Finally, Parseval's identity (2) can be deduced from (1), the continuity of the norm and orthonormality:

$$\|x\|_H^2 = \lim_{N \to \infty} \|s_N\|_X^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2.$$

□

Exercise 9.5 Let $(H, (\cdot, \cdot)_H)$ be a real Hilbert space.

(i) Prove that if the sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ converges weakly to x and $\|x_n\|_H \to \|x\|_H$, then it converges strongly to x.

(ii) Suppose $(x_n)_{n \in \mathbb{N}}$ converges weakly to x and $(y_n)_{n \in \mathbb{N}} \subseteq X$ converges strongly to y. Prove that $(x_n, y_n)_H \to (x, y)_H$.

(iii) Suppose $x \in H$ with $\|x\|_H \leq 1$, prove that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in H satisfying $\|x_n\|_H = 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{\text{w}} x$ as $n \to \infty$.

(iv) Prove the *Riemann-Lebesgue Lemma*: Let $f_n \colon [0, 2\pi] \to \mathbb{R}$ given by $f_n(t) = \sin(nt)$ for $n \in \mathbb{N}$, then $f_n \xrightarrow{\text{w}} 0$ in $L^2((0, 2\pi), \mathbb{R})$ as $n \to \infty$.

Solution. (i) Since $(x, \cdot)_H \in H^*$, weak convergence implies $(x, x_n)_H \to (x, x)_H = \|x\|_H^2$ as $n \to \infty$ and we have

$$\|x_n - x\|_H^2 = (x_n - x, x_n - x)_H = \|x_n\|_H^2 - 2(x, x_n)_H + \|x\|_H^2 \xrightarrow{n \to \infty} 0.$$

(ii) Weak convergence $x_n \overset{w}{\rightharpoonup} x$ implies that $(x_n, y)_H \to (x, y)_H$ as $n \to \infty$ and that $\|x_n\|_H \leq C$ for some constant $C > 0$ uniformly in n. Thus,

$$\begin{aligned}\Big|(x_n, y_n)_H - (x, y)_H\Big| &= \Big|(x_n, y_n - y)_H + (x_n, y)_H - (x, y)_H\Big| \\ &\leq C\|y_n - y\|_H + \Big|(x_n, y)_H - (x, y)_H\Big| \xrightarrow{n\to\infty} 0.\end{aligned}$$

(iii) If $x = 0$, then any orthonormal system converges weakly to x by Exercise 9.4. If $x \neq 0$, then an orthonormal system $(e_n)_{n\in\mathbb{N}}$ of H with $e_1 = x/\|x\|_H$ can be constructed via the Gram-Schmidt algorithm. For $n \in \mathbb{N}$, let

$$x_n := x + \Big(\sqrt{1 - \|x\|_H^2}\Big)e_{n+1}$$

Then, since $x \perp e_{n+1}$, we have $\|x_n\|^2 = \|x\|_H^2 + (1 - \|x\|_H^2) = 1$ for every $n \in \mathbb{N}$, and since $e_{n+1} \overset{w}{\rightharpoonup} 0$ by Exercise 9.4 (i), $x_n \overset{w}{\rightharpoonup} x$.

(iv) Let $f_n \colon [0, 2\pi] \to \mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for $n \in \mathbb{N}$. Then, since for any $m, n \in \mathbb{N}$ we have

$$\begin{aligned}\int_0^{2\pi} \sin(mt)\sin(nt)\,\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi} \cos\big((m-n)t\big) - \cos\big((m+n)t\big)\,\mathrm{d}t \\ &= \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n, \end{cases}\end{aligned}$$

we deduce that $(f_n/\sqrt{\pi})_{n\in\mathbb{N}}$ is an orthonormal system for $L^2([0, 2\pi])$, so by Exercise 9.4 (i) $f_n \overset{w}{\rightharpoonup} 0$ as $n \to \infty$.

□

Hints to Exercises.

9.1 For (i), one inequality follows from Hahn-Banach; for the other one argue by contradiction.

9.4 Use (after proving it) *Bessel's inequality*: $\sum_{n=0}^{\infty}|(x,e_n)_H|^2 \leq \|x\|_H^2$.

9.5 For (iii), use Exercise 9.4. Recall that in every Hilbert space the Graham-Schmidt process allows for construction of orthonormal systems.