Exercise 11.1 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We denote by

$$K(X,Y) = \{T \in L(X,Y) \mid \overline{T(B_1(0))} \subset Y \text{ compact}\}$$

the set of compact operators between X and Y. Prove the following statements.

(i) $T \in L(X,Y)$ is a compact operator if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in Y.

(ii) If $(Y, \|\cdot\|_Y)$ is complete, then $K(X,Y)$ is a closed subspace of $L(X,Y)$.

(iii) Let $T \in L(X,Y)$. If its range $T(X) \subset Y$ is finite-dimensional, then $T \in K(X,Y)$.

(iv) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If T or S is a compact operator, then $S \circ T$ is a compact operator.

(v) If X is reflexive, then any operator $T \in L(X,Y)$ which maps weakly convergent sequences to strongly convergent sequences, that is

$$x_n \xrightarrow{w} x \text{ in } X \quad \Longrightarrow \quad Tx_n \to x \text{ in } Y,$$

is a compact operator.

Solution. (i) "($\Rightarrow$)": Let $T \in L(X,Y)$ be a compact operator. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Then there exists $M > 0$ such that $\|x_n\|_X < M$ for all $n \in \mathbb{N}$. In particular, $\frac{1}{M}x_n \in B_1(0) \subset X$ and $\frac{1}{M}Tx_n \in T(B_1(0))$ for every $n \in \mathbb{N}$. Since $\overline{T(B_1(0))} \subset Y$ is compact, a subsequence $(\frac{1}{M}Tx_{n_k})_{k\in\mathbb{N}}$ converges in Y. Hence, $(Tx_{n_k})_{k\in\mathbb{N}}$ is also a convergent sequence.

"($\Leftarrow$)": Conversely, let $(y_n)_{n\in\mathbb{N}}$ be any sequence in $\overline{T(B_1(0))}$. For every $n \in \mathbb{N}$ there exists $y'_n \in T(B_1(0))$ such that $\|y_n - y'_n\|_Y \leq \frac{1}{n}$. Since there exists a sequence $(x'_n)_{n\in\mathbb{N}}$ in $B_1(0) \subset X$ such that $Tx'_n = y'_n$, a subsequence $y'_{n_k} \to y$ converges in Y as $k \to \infty$ by assumption. Since

$$\|y_{n_k} - y\|_Y \leq \|y_{n_k} - y'_{n_k}\| + \|y'_{n_k} - y\|_Y \to 0 \quad \text{as } k \to \infty,$$

we conclude that a subsequence of $(y_n)_{n\in\mathbb{N}}$ converges. Being closed, $\overline{T(B_1(0))}$ must contain the limit y which proves that $\overline{T(B_1(0))}$ is compact, i.e. T is a compact operator.

(ii) Part (ii) and linearity of the limit imply that the set of compact operators $K(X,Y) \subset L(X,Y)$ is a linear subspace. To prove that this subspace is closed, let $(T_k)_{k\in\mathbb{N}}$ be a sequence in $K(X,Y)$ such that $\|T_k - T\|_{L(X,Y)} \to 0$ for some $T \in L(X,Y)$ as $k \to \infty$. To show $T \in K(X,Y)$, consider a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X and choose the nested, unbounded subsets $\mathbb{N} \supset \Lambda_1 \supseteq \Lambda_2 \supseteq \ldots$ such that $(T_k x_n)_{n\in\Lambda_k}$ is convergent in Y with limit point $y_k \in Y$. This is possible by (i) since T_k is a compact operator for every $k \in \mathbb{N}$. Let $\Lambda \subset \mathbb{N}$ be the corresponding diagonal sequence (i. e. the k-th number in Λ is the k-th number in Λ_k). By continuity of $\|\cdot\|_Y$, we can estimate

$$\|y_k - y_m\|_Y = \lim_{\Lambda\ni n\to\infty} \|T_k x_n - T_m x_n\|_Y \leq \|T_k - T_m\|_{L(X,Y)} \sup_{n\in\Lambda} \|x_n\|_X$$

for any $k, m \in \mathbb{N}$. Since $(T_k)_{k\in\mathbb{N}}$ is convergent in $L(X,Y)$, we conclude that $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in Y. Since $(Y, \|\cdot\|_Y)$ is assumed to be complete, $y_k \to y$ for some $y \in Y$ as $k \to \infty$. It then suffices to prove the following.

Claim. The sequence $(Tx_n)_{n\in\Lambda}$ converges to y.

Proof. Let $\varepsilon > 0$. Choose a fixed $\kappa \in \mathbb{N}$ such that

$$\|T - T_\kappa\|_{L(X,Y)} < \varepsilon \qquad\qquad \|y_\kappa - y\|_Y \leq \varepsilon.$$

Since $T_\kappa x_n \to y_\kappa$ as $\Lambda \ni n \to \infty$, there exists $N \in \Lambda$ such that for every $\Lambda \ni n \geq N$ $\|T_\kappa x_n - y_\kappa\| \leq \frac{\varepsilon}{3}$. Finally, the claim follows from the estimate

$$\begin{aligned}
\|Tx_n - y\|_Y &\leq \|Tx_n - T_\kappa x_n\|_Y + \|T_\kappa x_n - y_\kappa\|_Y + \|y_\kappa - y\|_Y \\
&\leq \|T - T_\kappa\|_{L(X,Y)} \sup_{n\in\Lambda} \|x_n\|_X + \|T_\kappa x_n - y_\kappa\|_Y + \|y_\kappa - y\|_Y \\
&< 3\varepsilon \sup_{n\in\Lambda} \|x_n\|_X.
\end{aligned}$$

which holds for every $\Lambda \ni n \geq N$. Since ε is arbitrary, the claim follows. □

(iii) The image of $B_1(0)$ under $T \in L(X,Y)$ is bounded. If $T(X) \subset Y$ is of finite dimension, then so is so is $\overline{T(X)}$, and $\overline{T(B_1(0))}$ is compact as a bounded, closed subset of $\overline{T(X)}$.

(iv) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. Let $(x_n)_{n\in\mathbb{N}}$ be any bounded sequence in X.

Suppose T is a compact operator. Then, a subsequence $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in Y by (i). Since S is continuous, $(STx_{n_k})_{k\in\mathbb{N}}$ is convergent in Z, which by (i) proves that $S \circ T$ is a compact operator.

Suppose S is a compact operator. Since T is continuous, the sequence $(Tx_n)_{n\in\mathbb{N}}$ is bounded in Y. Then, a subsequence $(STx_{n_k})_{k\in\mathbb{N}}$ is convergent in Z by (i), which again proves that $S \circ T$ is a compact operator.

(v) Let $(x_n)_{n\in\mathbb{N}}$ be any bounded sequence in X. Since X is reflexive, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges weakly in X by the Eberlein–Šmulian theorem. Then, $(Tx_{n_k})_{k\in\mathbb{N}}$ is norm-convergent in Y by assumption and (i) implies that T is a compact operator. □

Exercise 11.2 Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C^1([0,1],\mathbb{R})$ so that, for every $n \in \mathbb{N}$,

$$f_n(0) = a \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|f_n'\|_{L^\infty((0,1))} \leq C,$$

for some a and C. Show that $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

Solution. For every $n \in \mathbb{N}$ and $x \in [0,1]$, we have

$$|f_n(x)| \leq |f_n(0)| + \int_0^x |f_n'(t)| dt \leq |a| + C,$$

Consequently, $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^0([0,1],\mathbb{R})$. It is also equicontinuous:

$$|f_n(x) - f_n(y)| = \left| \int_y^x f_n'(t)\ dt \right| \leq C|x-y|,$$

By the Arzelà–Ascoli theorem, $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence. □

Exercise 11.3 Let $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^m$ be a bounded open subset. Given $k \in L^2(\Omega \times \Omega, \mathbb{C})$, consider the linear operator $K\colon L^2(\Omega) \to L^2(\Omega,\mathbb{C})$ defined by

$$(Kf)(x) = \int_\Omega k(x,y) f(y)\, dy$$

(i) Prove that K is well-defined, i. e. $Kf \in L^2(\Omega,\mathbb{C})$ for any $f \in L^2(\Omega,\mathbb{C})$.

(ii) Prove that K is a compact operator.

(iii) Find an explicit expression for the adjoint $K^* : L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ (recall that for complex-valued function, the scalar product in $L^2(\Omega,\mathbb{C})$ is $(f,g)_{L^2} = \int_\Omega f\overline{g}dx$).

Solution. (i) Let $f \in L^2(\Omega, \mathbb{C})$. Then Hölder's inequality and Tonelli's theorem imply

$$\begin{aligned}\int_\Omega |(Kf)(x)|^2\,dx &= \int_\Omega \left|\int_\Omega k(x,y)f(y)\,dy\right|^2 dx \le \int_\Omega \left(\int_\Omega |k(x,y)f(y)|\,dy\right)^2 dx \\ &\le \int_\Omega \left(\int_\Omega |k(x,y)|^2\,dy\right)\|f\|^2_{L^2(\Omega)}\,dx = \|k\|^2_{L^2(\Omega\times\Omega)}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Since $k \in L^2(\Omega\times\Omega, \mathbb{C})$ by assumption, $\|Kf\|_{L^2(\Omega)} \le \|k\|_{L^2(\Omega\times\Omega)}\|f\|_{L^2(\Omega)} < \infty$ follows.

(ii) Being a Hilbert space, $L^2(\Omega)$ is reflexive. Exercise 11.1 (e) implies that $K\colon L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ is a compact operator if K maps weakly convergent sequences to norm-convergent sequences.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega,\mathbb{C})$ such that $f_n \xrightarrow{w} f$ as $n\to\infty$ for some $f \in L^2(\Omega,\mathbb{C})$. Since $k \in L^2(\Omega\times\Omega,\mathbb{C})$, Fubini's theorem implies that $k(x,\cdot) \in L^2(\Omega,\mathbb{C})$ for almost every $x\in\Omega$. Weak convergence therefore implies

$$(Kf_n)(x) = \Big\langle k(x,\cdot), f_n\Big\rangle_{L^2(\Omega)} \xrightarrow{n\to\infty} \Big\langle k(x,\cdot), f\Big\rangle_{L^2(\Omega)} = (Kf)(x)$$

for almost every $x\in\Omega$. As weakly convergent sequence, $(f_n)_{n\in\mathbb{N}}$ is bounded: there exists $C\in\mathbb{R}$ such that $\|f_n\|_{L^2(\Omega)} \le C$ for every $n\in\mathbb{N}$. By Hölder's inequality,

$$|(Kf_n)(x)| \le \int_\Omega |k(x,y)f_n(y)|\,dy \le \|k(x,\cdot)\|_{L^2(\Omega)}\|f_n\|_{L^2(\Omega)} \le C\|k(x,\cdot)\|_{L^2(\Omega)}.$$

The assumption $k\in L^2(\Omega\times\Omega)$ and Fubini's theorem imply that the function $x\mapsto C\|k(x,\cdot)\|_{L^2(\Omega)}$ is in $L^2(\Omega,\mathbb{C})$. Thus, $(Kf_n)(x)$ is dominated by a function in $L^2(\Omega,\mathbb{C})$. Since $(Kf_n)(x)$ converges pointwise for almost every $x\in\Omega$ to a function in $L^2(\Omega,\mathbb{C})$, the dominated convergence theorem implies L^2-convergence $\|Kf_n - Kf\|_{L^2(\Omega)} \to 0$.

(iii) For $f,g\in L^2(\Omega,\mathbb{C})$ using repeatedly Fubini's theorem we compute:

$$\begin{aligned}(Kf,g)_{L^2} &= \int_\Omega Kf(x)\overline{g(x)}dx \\ &= \int_\Omega \left(\int_\Omega k(x,y)f(y)dy\right)\overline{g(x)}dx \\ &= \int_{\Omega\times\Omega} k(x,y)\overline{g(x)}f(y)dxdy \\ &= \int_\Omega f(y)\left(\int_\Omega k(x,y)\overline{g(x)}dx\right)dy \\ &= \int_\Omega f(y)\overline{\left(\int_\Omega \overline{k(x,y)}g(x)dx\right)}dy = (f, K^*g)_{L^2},\end{aligned}$$

that is,

$$K^*g(x) = \int_\Omega \overline{k(y,x)} g(y) dy.$$

□

Exercise 11.4 Let $\ell^p_{\mathbb{C}}$ denote the space of $\mathbb{C}$-valued sequences of summable p-th power, namely

$$\ell^p_{\mathbb{C}} := \Big\{ x : \mathbb{N} \to \mathbb{C} \,:\, \sum_{n\in\mathbb{N}} |x_n|^p < \infty \Big\},$$

where as usual we write $x_n = x(n)$. The space is endowed with its standard Banach norm $\|\cdot\|_{\ell^p_{\mathbb{C}}}$. Given $a \in \ell^\infty_{\mathbb{C}}$ we define the operator $T\colon \ell^2_{\mathbb{C}} \to \ell^2_{\mathbb{C}}$ by $(Tx)_n = a_n x_n$.

(i) Prove that $T \in L(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ and compute its operator norm.

(ii) Prove that T is self-adjoint if and only if $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

(iii) Prove that T is compact if and only if $\lim\limits_{n\to\infty} a_n = 0$.

Solution. (i) Given $a \in \ell^\infty_{\mathbb{C}}$, let $(Tx)_n = a_n x_n$ for $x \in \ell^2_{\mathbb{C}}$. We obtain $\|T\| \le \|a\|_{\ell^\infty_{\mathbb{C}}}$ from

$$\|Tx\|^2_{\ell^2_{\mathbb{C}}} = \sum_{n\in\mathbb{N}} |a_n x_n|^2 \le \|a\|^2_{\ell^\infty_{\mathbb{C}}} \|x\|^2_{\ell^2_{\mathbb{C}}}.$$

Given any $k \in \mathbb{N}$ let $e_k = (0,\ldots,0,1,0,\ldots) \in \ell^2_{\mathbb{C}}$, where the 1 is at k-th position. Then, $\|Te_k\|_{\ell^2_{\mathbb{C}}} = |a_k| = |a_k| \|e_k\|_{\ell^2_{\mathbb{C}}}$ implies $\|T\| \ge |a_k|$. Since $k \in \mathbb{N}$ is arbitrary, $\|T\| \ge \|a\|_{\ell^\infty_{\mathbb{C}}}$ follows. Hence $\|T\| = \|a\|_{\ell^\infty_{\mathbb{C}}}$.

(ii) The adjoint operator T^* of T is given by $(T^*y)_n = \overline{a_n} y_n$ for $y \in \ell^2_{\mathbb{C}}$ because

$$\forall x, y \in \ell^2_{\mathbb{C}} \qquad (x, T^*y)_{\ell^2_{\mathbb{C}}} = (Tx, y)_{\ell^2_{\mathbb{C}}} = \sum_{n\in\mathbb{N}} a_n x_n \overline{y_n} = \sum_{n\in\mathbb{N}} x_n \overline{\overline{a_n} y_n}.$$

and we conclude $T = T^* \;\Leftrightarrow\; a_n = \overline{a_n} \;\; \forall n \in \mathbb{N}$.

(iii) Let $T \in L(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ and $e_k \in \ell^2_{\mathbb{C}}$ be as in (i). Being an orthonormal system of the Hilbert space $\ell^2_{\mathbb{C}}$, the sequence $(e_n)_{n\in\mathbb{N}}$ converges weakly to zero. If T is a compact operator, then $|a_n| = \|Te_n\|_{\ell^2_{\mathbb{C}}} \to 0$ as $n \to \infty$ as a consequence of Exercise 11.1 (v).

Conversely, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $a_n \to 0$ as $n \to \infty$ and let $T \in L(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ be the corresponding multiplication operator. Let $(x_k)_{k\in\mathbb{N}}$,

$x_k = (x_{k,n})_{n\in\mathbb{N}}$ be any bounded sequence in $\ell^2_{\mathbb{C}}$ and $C > 0$ a constant such that $\|x_k\|_{\ell^2_{\mathbb{C}}} \leq C$ for every $k \in \mathbb{N}$. Since $\ell^2_{\mathbb{C}}$ is reflexive, there exists $x \in \ell^2_{\mathbb{C}}$ so that, after possibly passing to a subsequence, $x_k \xrightarrow{\text{w}} x$. In particular,

$$\lim_{k\to\infty} x_{k,n} = \lim_{k\to\infty} (e_n, x_k)_{\ell^2_{\mathbb{C}}} = (e_n, x)_{\ell^2_{\mathbb{C}}} = x_n. \tag{$*$}$$

Moreover, since $B_C(0;\ell^2_{\mathbb{C}})$ is weakly closed, $\|x\|_{\ell^2_{\mathbb{C}}} \leq C$. Let $\varepsilon > 0$. By assumption, there exists $N \in \mathbb{N}$ such that $|a_n|^2 < \varepsilon$ for all $n \geq N$. Then

$$\begin{aligned}
&\limsup_{k\to\infty} \left\|Tx_k - Tx\right\|^2_{\ell^2_{\mathbb{C}}} \\
&\leq \limsup_{k\to\infty} \sum_{n=1}^{N} |a_n(x_{k,n} - x_n)|^2 + \limsup_{k\to\infty} \sum_{n=N+1}^{\infty} |a_n(x_{k,n} - x_n)|^2 \\
&+ \varepsilon \limsup_{k\to\infty} \varepsilon \sum_{n\in\mathbb{N}} |(x_{k,n} - x_n)|^2 \\
&\leq 2C\varepsilon.
\end{aligned}$$

Thus, since ε is arbitrary, a subsequence of $(Tx_k)_{k\in\Lambda}$ converges in $\ell^2_{\mathbb{C}}$, which proves that T is a compact operator. $\square$

Hints to Exercises.

11.1 For (v), use Eberlein-Šmulian's Theorem.

11.3 Use repeatedly the theorem of Fubini-Tonelli. For (ii) Use Exercise 10.1 (v).