Exercise 12.1 The right shift map on the space ℓ^2 is given by

$$\begin{aligned} S\colon \ell^2 &\to \ell^2 \\ (x_1, x_2, \dots) &\mapsto (0, x_1, x_2, \dots). \end{aligned}$$

(i) Show that that the map S is a continuous linear operator with norm $\|S\| = 1$.

(ii) Compute the eigenvalues and the spectral radius of S.

(iii) Show that S has a left inverse which is not a right inverse, i.e. there exists $T\colon \ell^2 \to \ell^2$ with $T \circ S = \mathrm{id}_{\ell^2}$ but $S \circ T \neq \mathrm{id}_{\ell^2}$. Is it possible to find a right inverse of S, i.e. $Q : \ell^2 \to \ell^2$ so that $S \circ Q = \mathrm{id}_{\ell^2}$?

Solution. (i) Let $x \in \ell^2$. By definition of S and the ℓ^2-norm $\|Sx\|_{\ell^2} = \|x\|_{\ell^2}$, which implies $\|S\| = 1$. Being linear and bounded, the map S is continuous.

(ii) Suppose $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$ satisfies $Sx = \lambda x$ for some $\lambda \in \mathbb{R}$. Then

$$(0, x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3 \dots).$$

If $\lambda = 0$, then $x = 0$ is immediate. If $\lambda \neq 0$, then $x = 0$ follows via

$$0 = \lambda x_1 \;\Rightarrow\; 0 = x_1 = \lambda x_2 \;\Rightarrow\; 0 = x_2 = \lambda x_3 \;\Rightarrow\; \dots$$

We conclude that S does not have eigenvalues. Since $\|S^n\| = 1$ for every $n \in \mathbb{N}$ (the proof is as in (i)), the spectral radius of S is

$$r_S = \lim_{n\to\infty} \|S^n\|^{\frac{1}{n}} = 1.$$

(iii) We define $T\colon \ell^2 \to \ell^2$ to be the left shift map $T\colon (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. Then, $T \circ S = \mathrm{id}_{\ell^2}$ and $S \circ T \neq \mathrm{id}_{\ell^2}$. Indeed,

$$\begin{aligned} (T \circ S)(x_1, x_2, \dots) &= T(0, x_1, x_2, \dots) = (x_1, x_2, \dots), \\ (S \circ T)(x_1, x_2, \dots) &= S(x_2, x_3, \dots) = (0, x_2, x_3, \dots). \end{aligned}$$

It is never possible find a right inverse for S: this would be equivalent to say that S is bijective, which is clearly false since $(1, 0, 0, \dots) \notin S(\ell^2)$. □

Exercise 12.2 Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. Recall two definitions:

- A linear operator $T \in L(H)$ is called an *isometry* if $\|Tx\|_H = \|x\|_H$ for every $x \in H$;
- An invertible linear operator $T \in L(H)$ is *unitary* if $T^* = T^{-1}$.

With these definitions,

(i) Prove that T is an isometry if and only if it preserves the scalar product, that is

$$\langle Tx, Ty\rangle_H = \langle x, y\rangle_H \quad \text{for every } x, y \in H.$$

(ii) Prove that $T \in L(H)$ is unitary if and only if T is a bijective isometry.

(iii) Prove that if $T \in L(H)$ is unitary, then its spectrum lies on the unit circle:

$$\sigma(T) \subset \mathbb{S}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Solution. (i) The sufficiency of the condition is obvious; as for the necessity, it suffices to use the the complex polarization identity

$$\langle x, y\rangle_H = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) + \frac{i}{4}\Big(\|x+iy\|_H^2 - \|x-iy\|_H^2\Big)$$

(the proof is similar to that for the parallelogram identity) to conclude $\langle Tx, Ty\rangle_H = \langle x, y\rangle_H$ for every $x, y \in H$.

(ii) If $T \in L(H)$ is unitary, then T is invertible with inverse $T^{-1} = T^* \in L(H)$. T is also an isometry, because for every $x \in H$ we have

$$\|Tx\|_H^2 = \langle Tx, Tx\rangle_H = \langle T^*Tx, x\rangle_H = \langle x, x\rangle_H = \|x\|_H^2.$$

Conversely suppose $T \in L(H)$ is an bijective isometry. Then by (i),

$$\langle T^*Tx, y\rangle_H = \langle Tx, Ty\rangle_H = \langle x, y\rangle_H$$

for every $x, y \in H$ which implies $T^*Tx = x$ for every $x \in H$. Since T is bijective, we obtain $T^* = T^{-1}$ which means that T is unitary .

(iii) Let $T \in L(H)$ be unitary. Part (ii) implies that T and $T^* = T^{-1}$ are bijective isometries. Therefore, $\|T\| = 1 = \|T^*\|$. Since the spectral radius of T is bounded from above by $\|T\| = 1$, we obtain $\{\lambda \in \mathbb{C} \mid |\lambda| > 1\} \subset \rho(T)$ (Satz 6.5.3).

Given $\lambda \in \mathbb{C}$ with $0 \le |\lambda| < 1$, the spectral radius of the operator (λT^*) is bounded from above by $\|\lambda T^*\| = |\lambda| < 1$. Thus, $(1 - \lambda T^*)$ is invertible on H by Satz 2.2.7. Hence, $(\lambda - T) = -T \circ (1 - \lambda T^*)$ is bijective as composition of bijective operators and we obtain $\lambda \in \rho(T)$. To conclude, $\sigma(T) \subset \mathbb{S}^1$. $\square$

Exercise 12.3 Let $\Omega \subset \mathbb{R}^m$ be an open bounded subset. Given $k \in L^2(\Omega \times \Omega, \mathbb{C})$ such that $k(x,y) = \overline{k(y,x)}$ for almost every $(x,y) \in \Omega \times \Omega$, consider the operator $K\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ defined by

$$(Kf)(x) = \int_\Omega k(x,y) f(y)\, dy,$$

and the operator $A\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ defined by

$$A(f)(x) = f(x) - Kf(x).$$

Prove that injectivity of A and surjectivity of A are equivalent.

Solution. From Exercise 11.3 (iii), the operator K is self-adjoint. Therefore, the operator $A = (1 - K)\colon L^2(\Omega) \to L^2(\Omega)$ is also self-adjoint (Beispiel 6.4.2 (ii)).

According to Exercise 11.3 (ii), K is a compact operator, which implies that the operator $A = (1 - K)$ has closed image $\operatorname{im}(A) \subset H$. According to Banach's closed range theorem, this is equivalent to $\operatorname{im}(A) = \ker(A^*)^\perp$. Since $A^* = A$, we conclude

$$A \text{ surjective } \Leftrightarrow H = \operatorname{im}(A) = \ker(A)^\perp \Leftrightarrow \ker(A) = \{0\} \Leftrightarrow A \text{ injective.}$$

$\square$

Exercise 12.4 Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space over $\mathbb{C}$.

(i) Let $A \in L(H)$ be a self-adjoint operator and let $\lambda \in \rho(A)$ be an element in its resolvent set. Show that the resolvent $R_\lambda := (\lambda - A)^{-1}$ is a normal operator, that is $R_\lambda R_\lambda^* = R_\lambda^* R_\lambda$.

(ii) Let $A, B \in L(H)$ be self-adjoint operators. The *Hausdorff distance* of their spectra $\sigma(A), \sigma(B) \subset \mathbb{C}$ is defined to be

$$d\Big(\sigma(A), \sigma(B)\Big) := \max\left\{ \sup_{\alpha \in \sigma(A)} \left(\inf_{\beta \in \sigma(B)} |\alpha - \beta| \right), \sup_{\beta \in \sigma(B)} \left(\inf_{\alpha \in \sigma(A)} |\alpha - \beta| \right) \right\}.$$

Prove that

$$d\Big(\sigma(A), \sigma(B)\Big) \leq \|A - B\|_{L(H)}.$$

Remark. The Hausdorff distance d is in fact a distance on compact subsets of $\mathbb{C}$. In particular, it restricts to an actual distance function on the spectra of bounded linear operators.

Solution. (i) Given the self-adjoint operator $A \in L(H)$ and an element $\lambda \in \rho(A)$, the operator $(\lambda - A) \in L(H)$ is bijective with inverse $R_\lambda = (\lambda - A)^{-1} \in L(H)$. Exercise 10.4 (i) then implies that R_λ^* is bijective and according to Exercise 10.3 (iii), which also holds for the adjoint operator instead of the dual operator, there holds,

$$R_\lambda^* = \Big((\lambda - A)^{-1}\Big)^* = \Big((\lambda - A)^*\Big)^{-1} = (\overline{\lambda} - A^*)^{-1} = (\overline{\lambda} - A)^{-1} = R_{\overline{\lambda}}.$$

Alternatively, for any $x, y \in H$, we can directly compute

$$\begin{aligned}\langle x, y\rangle_H &= \langle(\lambda - A)R_\lambda x, y\rangle_H = \langle \lambda R_\lambda x, y\rangle_H - \langle AR_\lambda x, y\rangle_H \\ &= \langle R_\lambda x, \overline{\lambda} y\rangle_H - \langle R_\lambda x, Ay\rangle_H = \langle R_\lambda x, (\overline{\lambda} - A)y\rangle_H = \langle x, R_\lambda^*(\overline{\lambda} - A)y\rangle_H\end{aligned}$$

which implies $R_\lambda^*(\overline{\lambda} - A)y = y$ for any $y \in H$. According to Satz 6.5.2, resolvents commute: $R_\lambda R_{\overline{\lambda}} = R_{\overline{\lambda}} R_\lambda$. This implies that R_λ is a normal operator.

(ii) Let $A, B \in L(H)$ be self-adjoint operators. By symmetry of the Hausdorff distance (in the sense that we can switch the roles of A and B), it suffices to prove

$$\sup_{\alpha \in \sigma(A)} \Big(\inf_{\beta \in \sigma(B)} |\alpha - \beta| \Big) \leq \|A - B\|_{L(H)}.$$

The claim follows if we show the following implication for any $\alpha \in \mathbb{C}$:

$$\inf_{\beta \in \sigma(B)} |\alpha - \beta| > \|A - B\|_{L(H)} \qquad \Rightarrow \ \alpha \in \rho(A) = \mathbb{C} \setminus \sigma(A).$$

Let $\alpha \in \mathbb{C}$ satisfy $\inf_{\beta \in \sigma(B)} |\alpha - \beta| > \|A - B\|_{L(H)}$. Since the claim is trivial otherwise, we may assume $\|A - B\|_{L(H)} > 0$. Then, α has positive distance from $\sigma(B)$ which implies $\alpha \in \rho(B)$. Hence, $(\alpha - B)^{-1}$ is well-defined and we obtain

$$(\alpha - A) = (\alpha - B) - (A - B) = \Big(1 - (A - B)(\alpha - B)^{-1}\Big)(\alpha - B). \qquad (*)$$

Since $(\alpha - B)$ is bijective, it remains to prove that $\Big(1 - (A - B)(\alpha - B)^{-1}\Big)$ is bijective. This follows from Satz 2.2.7 if we prove $\|(A - B)(\alpha - B)^{-1}\|_{L(H)} < 1$.

Consider the rational function $f_\alpha\colon \mathbb{C} \to \mathbb{C}$ given by $f_\alpha(z) = (\alpha - z)^{-1}$. By assumption,

$$\frac{1}{\|A - B\|} > \frac{1}{\inf\limits_{\beta \in \sigma(B)} |\alpha - \beta|} = \sup_{\beta \in \sigma(B)} \frac{1}{|\alpha - \beta|} = \sup\Big\{|x| \ \Big| \ x \in f_\alpha(\sigma(B))\Big\}.$$

The spectral mapping theorem (Satz 6.5.4) implies $f_\alpha(\sigma(B)) = \sigma(f_\alpha(B))$. Thus,

$$\frac{1}{\|A-B\|} > \sup\Big\{|x| \;\Big|\; x \in \sigma(f_\alpha(B))\Big\} = \sup_{x\in\sigma(f_\alpha(B))} |x| = r_{f_\alpha(B)} \tag{$\dagger$}$$

where we use the characterisation of spectral radius proven in Satz 6.5.3. Since $f_\alpha(B) = (\alpha - B)^{-1} =: R$ is a resolvent of B, it is a normal operator by (i). Hence,

$$\begin{aligned}
&\|Rx\|_H^2 = \langle Rx, Rx\rangle_H = \langle R^*Rx, x\rangle_H = \langle RR^*x, x\rangle_H = \langle R^*x, R^*x\rangle_H = \|R^*x\|_H^2,\\
&\|Rx\|_H^2 = \langle R^*Rx, x\rangle_H \le \|R^*Rx\|_H\|x\|_H \le \|R^*R\|\|x\|_H^2,\\
&\Rightarrow \|R\|^2 \le \|R^*R\| \le \|R^*\|\|R\| = \|R\|^2,\\
&\Rightarrow \|R\|^2 = \|R^*R\| = \sup_{\|x\|_H=1}\|R^*(Rx)\|_H = \sup_{\|x\|_H=1}\|R(Rx)\|_H = \|R^2\|.
\end{aligned}$$

(Note how the last identity makes use of the first identity.) Inductively, we obtain $\|R\|^{2^n} = \|R^{2^n}\|$ for every $n \in \mathbb{N}$ which implies $r_{f_\alpha(B)} = r_R = \|R\| = \|(\alpha - B)^{-1}\|$. Combined with estimate ($\dagger$), we obtain $\frac{1}{\|A-B\|} > \|(\alpha - B)^{-1}\|$, which yields

$$\|(A-B)(\alpha - B)^{-1}\| \le \|A-B\|\|(\alpha-B)^{-1}\| < 1$$

and proves the claim: From ($*$) we conclude $\alpha \in \rho(A)$. □

Exercise 12.5 *(Heisenberg's Uncertainty Principle)* Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. Let $D_A, D_B \subset H$ be dense subspaces and let $A\colon D_A \subset H \to H$ and $B\colon D_B \subset H \to H$ be symmetric linear operators. Assume that

$$A(D_A \cap D_B) \subset D_B \quad \text{and} \quad B(D_A\cap D_B) \subset D_A,$$

and define the *commutator* of A and B as

$$[A,B]\colon D_{[A,B]} \subset H \to H, \qquad [A,B](x) \mapsto A(Bx) - B(Ax),$$

where $D_{[A,B]} := D_A \cap D_B$.

(i) Prove that

$$\Big|\langle x, [A,B]x\rangle_H\Big| \le 2\|Ax\|_H\|Bx\|_H \quad \text{for every } x \in D_{[A,B]}.$$

(ii) Define now the *standard deviation* of A

$$\varsigma(A,x) := \sqrt{\langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2}$$

at each $x \in D_A$ with $\|x\|_H = 1$. Verify that $\varsigma(A,x)$ is well-defined for every x (i.e. that the radicand is real and non-negative) and prove that for every $x \in D_{[A,B]}$ with $\|x\|_H = 1$ there holds

$$\left|\langle x, [A,B]x\rangle_H\right| \leq 2\varsigma(A,x)\,\varsigma(B,x).$$

Remark. The possible *states* of a quantum mechanical system are given by elements $x \in H$ with $\|x\|_H = 1$. Each *observable* is given by a symmetric linear operator $A\colon D_A \subset H \to H$. If the system is in state $x \in D_A$, we measure the observable A with uncertainty $\varsigma(A,x)$.

(iii) Let $A\colon D_A \to H$ and $B\colon D_B \to H$ be as above. A, B is called *Heisenberg pair* if

$$[A,B] = i\,\mathrm{Id}\,.$$

Show that, if A, B is a Heisenberg pair with B continuous (and $D_B = H$), then A cannot be continuous.

(iv) Consider the Hilbert space $(H, \langle\cdot,\cdot\rangle_H) = \Big(L^2([0,1],\mathbb{C}), \langle\cdot,\cdot\rangle_{L^2}\Big)$ and the subspace

$$C_0^1([0,1],\mathbb{C}) := \{f \in C^1([0,1],\mathbb{C}) \mid f(0) = 0 = f(1)\}.$$

Recall that $C_0^1([0,1],\mathbb{C}) \subset L^2([0,1],\mathbb{C})$ is a dense subspace. The operators

$$\begin{aligned} P\colon C_0^1([0,1],\mathbb{C}) &\to L^2([0,1],\mathbb{C}), & Q\colon L^2([0,1],\mathbb{C}) &\to L^2([0,1],\mathbb{C}) \\ f(s) &\mapsto if'(s) & f(s) &\mapsto sf(s) \end{aligned}$$

correspond to the observables *momentum* and *position*. Check that P and Q are well-defined, symmetric operators. Check that $[P,Q]\colon C_0^1([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C})$ is well-defined.

Show that P and Q form a Heisenberg pair and conclude that the *uncertainty principle* holds: for every $f \in C_0^1([0,1],\mathbb{C})$ with $\|f\|_{L^2([0,1],\mathbb{C})} = 1$ there holds

$$\varsigma(P,f)\,\varsigma(Q,f) \geq \tfrac{1}{2}.$$

Thus we conclude: *The more precisely the momentum of some particle is known, the less precisely its position can be known, and vice versa.*

Solution. (i) Let $x \in D_{[A,B]} := D_A \cap D_B$. Then, applying the Cauchy–Schwarz inequality,

$$\begin{aligned} \left|\langle x, [A,B]x\rangle_H\right| &\leq \left|\langle x, A(Bx)\rangle_H\right| + \left|\langle x, B(Ax)\rangle_H\right| \\ &= \left|\langle Ax, Bx\rangle_H\right| + \left|\langle Bx, Ax\rangle_H\right| \\ &\leq \|Ax\|_H\|Bx\|_H + \|Bx\|_H\|Ax\|_H \\ &= 2\|Ax\|_H\|Bx\|_H. \end{aligned}$$

(ii) Since A is a symmetric operator, $\langle x, Ax\rangle_H$ is real for every $x \in D_A \subset D_{A^*}$. Indeed,

$$\langle x, Ax\rangle_H = \langle A^*x, x\rangle_H = \langle Ax, x\rangle_H = \overline{\langle x, Ax\rangle}_H.$$

Moreover, for $x \in D_A$ with $\|x\|_H = 1$, we have

$$\langle x, Ax\rangle_H^2 \leq \|x\|_H^2 \|Ax\|_H^2 = \langle Ax, Ax\rangle_H$$

Therefore, the radicand in the definition of the standard deviation is real and $\varsigma(A, x)$ is well-defined. For any $\lambda, \mu \in \mathbb{R}$, the commutators $[A, B]$ and $[A - \lambda, B - \mu]$ agree:

$$\begin{aligned}[A - \lambda, B - \mu] &= (A - \lambda)(B - \mu) - (B - \mu)(A - \lambda) \\ &= AB - \mu A - \lambda B + \lambda\mu - BA + \lambda B + \mu A - \lambda\mu = [A, B].\end{aligned}$$

Since A is symmetric and $\lambda \in \mathbb{R}$, the operator $\tilde{A} = A - \lambda$ is also symmetric on $D_{\tilde{A}} = D_A$. Moreover, for any $x \in D_A$,

$$\begin{aligned}\|\tilde{A}x\|_H^2 = \langle \tilde{A}x, \tilde{A}x\rangle_H &= \langle Ax - \lambda x, Ax - \lambda x\rangle_H \\ &= \langle Ax, Ax\rangle_H - \lambda\langle x, Ax\rangle_H - \lambda\langle Ax, x\rangle_H + \lambda^2\langle x, x\rangle_H \\ &= \langle Ax, Ax\rangle_H - 2\lambda\langle x, Ax\rangle_H + \lambda^2\langle x, x\rangle_H.\end{aligned}$$

We observe that if we choose $\lambda = \langle x, Ax\rangle_H \in \mathbb{R}$ and if $\|x\|_H = 1$, then

$$\|\tilde{A}x\|_H^2 = \langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2 = \varsigma(A, x)^2.$$

Now, let $x \in D_{[A,B]} := D_A \cap D_B$ with $\|x\|_H = 1$ be arbitrary. Since the operators $\tilde{A} := A - \langle x, Ax\rangle_H$ and $\tilde{B} := B - \langle x, Bx\rangle_H$ are symmetric, part i applies and yields

$$\Big|\langle x, [A, B]x\rangle_H\Big| = \Big|\langle x, [\tilde{A}, \tilde{B}]x\rangle_H\Big| \leq 2\|\tilde{A}x\|_H\|\tilde{B}x\|_H = 2\varsigma(A, x)\,\varsigma(B, x).$$

(iii) Suppose, $B \in L(H)$ and $A\colon D_A \subset H \to H$ satisfy

$$[A, B] = i\,\mathrm{Id}\,.$$

By assumption, $D_{[A,B]} = D_A \cap H = D_A$ and $B(D_A) \subset D_A$. In particular, for any $n \in \mathbb{N}$ the inclusion $B^n(D_A) \subset D_A$ is satisfied, which is necessary to define $[A, B^n]$. We prove $[A, B^n] = niB^{n-1}$ by induction. For $n = 1$, the claim holds by assumption. Suppose, it is true for some $n \in \mathbb{N}$. Then

$$\begin{aligned}[A, B^{n+1}] &= AB^{n+1} - B^{n+1}A = \Big(AB^n - B^nA + B^nA\Big)B - B^{n+1}A \\ &= \Big([A, B^n] + B^nA\Big)B - B^{n+1}A = niB^{n-1}B + B^nAB - B^{n+1}A \\ &= niB^n + B^n[A, B] = niB^n + iB^n = (n + 1)iB^n.\end{aligned}$$

A consequence is that B cannot be nilpotent: If $B^n = 0$ for some $n \in \mathbb{N}$, then $B^{n-1} = \frac{1}{ni}[A, B^n] = 0$ which iterates to $B = 0$ in contradiction to $[A, B] \neq 0$. Suppose by contradiction that A has finite operator norm $\|A\|$. Then,

$$n\|B^{n-1}\| = \|[A, B^n]\| \leq \|AB^n\| + \|B^n A\| \leq 2\|A\|\|B^{n-1}\|\|B\|.$$

Since $\|B^{n-1}\| \neq 0$, we obtain $2\|A\| \geq \frac{n}{\|B\|}$ for every $n \in \mathbb{N}$, thus $\|A\|$ cannot be finite and the contradiction is reached.

(iv) If $f \in C^1([0,1];\mathbb{C})$, then f' is bounded and in particular $f' \in L^2([0,1];\mathbb{C})$. Therefore, the linear operators

$$\begin{aligned} P\colon C_0^1([0,1];\mathbb{C}) &\to L^2([0,1];\mathbb{C}), & \qquad Q\colon L^2([0,1];\mathbb{C}) &\to L^2([0,1];\mathbb{C}) \\ f(s) &\mapsto if'(s) & f(s) &\mapsto sf(s) \end{aligned}$$

are indeed well-defined. They are also symmetric. For Q this follows trivially from $s \in [0,1] \subset \mathbb{R}$. Given any $f, g \in D_P := C_0^1([0,1];\mathbb{C})$, we have

$$\langle Pf, g\rangle_{L^2} = \int_0^1 if'(s)\overline{g}(s)\,ds = -\int_0^1 if(s)\overline{g}'(s)\,ds = \int_0^1 f(s)\overline{ig'(s)}\,ds = \langle f, Pg\rangle_{L^2}.$$

When integrating by parts, the boundary terms vanish due to $f(0) = 0 = f(1)$. Hence, $P\colon C_0^1([0,1];\mathbb{C}) \to L^2([0,1];\mathbb{C})$ is symmetric (but *not* self-adjoint! see Beispiel 6.6.1).

Next, we verify that the commutator $[P, Q]$ is well-defined. Since $D_Q = L^2([0,1];\mathbb{C})$ is the whole space, the only thing to check is that $Qf\colon s \mapsto sf(s)$ is in $D_P = C_0^1([0,1];\mathbb{C})$ whenever $f \in D_{[P,Q]} = C_0^1([0,1];\mathbb{C})$. But this follows trivially from the product rule. Moreover,

$$([P,Q]f)(s) = (P(Qf))(s) - (Q(Pf))(s) = if(s) + isf'(s) - sif'(s) = if(s)$$

for almost every $s \in [0,1]$ which proves that P, Q is a Heisenberg-pair. By part ii,

$$\forall f \in C_0^1,\ \|f\|_{L^2} = 1: \quad \varsigma(P,f)\,\varsigma(Q,f) \geq \tfrac{1}{2}\Big|\langle f, [P,Q]f\rangle_{L^2}\Big| = \tfrac{1}{2}\Big|\langle f, if\rangle_{L^2}\Big| = \tfrac{1}{2}.$$

□

Hints to Exercises.

12.2 For (i), use the the complex polarization identity

$$\langle x, y\rangle_H = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) + \frac{i}{4}\Big(\|x+iy\|_H^2 - \|x-iy\|_H^2\Big).$$

For (iii), use Satz 6.5.3 and Satz 2.2.7.

12.3 Use Exercise 11.3.

12.4 Prove that $R_\lambda^* = R_{\overline{\lambda}}$ and use that resolvents to different values commute (Satz 6.5.2). Argue that it suffices to show the following implication for any $\alpha \in \mathbb{C}$:

$$\inf_{\beta\in\sigma(B)}|\alpha-\beta| > \|A-B\|_{L(H)} \qquad \Rightarrow \ \alpha \in \rho(A).$$

Given $f_\alpha(z) = (\alpha - z)^{-1}$, the spectral mapping theorem implies $f_\alpha(\sigma(B)) = \sigma(f_\alpha(B))$. Show that normal operators R have spectral radius $r_R = \|R\|$. Apply Satz 2.2.7.

12.5 For (ii): in order to apply (i), find symmetric operators $\tilde{A} = A-\lambda$ and $\tilde{B} = B-\mu$ satisfying

$$[A,B] = [\tilde{A},\tilde{B}], \qquad \varsigma(A,x) = \|\tilde{A}x\|_H, \qquad \varsigma(B,x) = \|\tilde{B}x\|_H.$$

For (iii), begin by checking that $[A, B^n]$ is well-defined and prove $[A, B^n] = niB^{n-1}$ for every $n \in \mathbb{N}$.