ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
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7 August 2018

Functional Analysis I
Repetition Exam

Name:
Student number:
Study program:

- Put your student identity card onto the desk.
- During the exam no written aids nor calculators or any other electronic device are allowed in the exam room. **Phones must be switched off and stowed away** in your bag during the whole duration of the exam.
- A4-paper is provided. No other paper is allowed. Write with blue or black pens. **Do *not* use pencils, erasable pens, red or green ink, nor Tipp-Ex.**
- Start every problem on a new sheet of paper and write your name on every sheet of paper. Leave enough ($\approx 3\,\mathrm{cm}$) empty space on the margins (top, bottom and sides). You can solve the problems in any order you want, but please sort them in the end.
- You will be asked to return **all** sheets of paper you are assigned, however you have the freedom to clearly cross those sheets you do not want us to consider during the grading process (i. e. scratch paper).
- Please write neatly! Please do not put the graders in the unpleasant situation of being incapable of reading your solutions, as this will certainly not play in your favour!
- All your answers do need to be properly justified. It is fine and allowed to use theorems/statements proved in class or in the homework (i. e. in problem sets 1–13) without reproving them (**unless otherwise stated**), but you should provide a precise statement of the result in question.
- Said $x_i \in \{0, \ldots, 10\}$ your score on exercise i, your grade will be bounded from below by $\min\{6, \frac{1}{10}\sum_{i=1}^{7} x_i\}$.
The *complete and correct* solution of 6 problems out of 7 is enough to obtain the maximum grade 6,0.
The *complete and correct* solution of 4 problems out of 7 is enough to obtain the pass/sufficient grade 4,0.
- The duration of the exam is **180 minutes**.

Do not fill out this table!

task	points	check
1	[10]	
2	[10]	
3	[10]	
4	[10]	
5	[10]	
6	[10]	
7	[10]	
total	[70]	

grade:

Problem 1. **[10 points]**

(a) Given two topological spaces (X_1, τ_1), (X_2, τ_2) define what it means that a function $f\colon X_1 \to X_2$ is open. Provide an explicit example of two topological spaces (X_1, τ_1), (X_2, τ_2) and a map $f\colon X_1 \to X_2$ that is injective, continuous but *not* open.

(b) State the open mapping theorem.

(c) Is it possible to find a linear bijection $T\colon \ell^1 \to \ell^\infty$ that is continuous? If so, provide an explicit example; otherwise, justify your answer with a proof.

Problem 2. **[10 points]**

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space over $\mathbb{R}$. Given a positive integer n, consider n pairwise distinct points $x_1, \ldots, x_n$ in X and the functional

$$F\colon X \to \mathbb{R}, \qquad\qquad F(x) = \sum_{i=1}^{n} \|x - x_i\|_X^2.$$

(a) Prove that the functional F has a global minimum on X, namely the value $\inf\limits_{x \in X} F(x)$ is a real number attained by F at some $\overline{x} \in X$.

Let us now specify the result above to the case when $(X, \|\cdot\|_X)$ is a Hilbert space (thus $\|\cdot\|_X$ is induced by a scalar product $\langle \cdot, \cdot \rangle_X$).

(b) Prove that the minimum $\overline{x} \in X$ is unique, and that $\overline{x}$ belongs to the convex hull K of $\{x_1, \ldots, x_n\}$.

Problem 3. **[10 points]**

(a) Let V be a vector space over $\mathbb{R}$ and let $d\colon V \times V \to \mathbb{R}$ be a distance. State necessary and sufficient conditions for $d(\cdot, \cdot)$ to be induced by a norm $\|\cdot\|$, in the sense that

$$d(v_1, v_2) = \|v_1 - v_2\| \quad \forall\, v_1, v_2 \in V.$$

(Note that only a statement is requested, no proof.)

(b) Consider the vector space $C([0, \infty[; \mathbb{R})$ consisting of continuous functions defined on $[0, \infty[\subset \mathbb{R}$ and attaining real values, and the distance

$$d(f_1, f_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f_1 - f_2\|_{C^0([0,n])}}{1 + \|f_1 - f_2\|_{C^0([0,n])}}$$

where $\|f\|_{C^0([0,n])} = \sup\limits_{x \in [0,n]} |f(x)|$. Is d induced by a norm?

Problem 4. **[10 points]**

Consider the space $(c_0, \|\cdot\|_{\ell^\infty})$, where as usual $c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty : \lim_{n\to\infty} x_n = 0\}$ and the subspace $c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty : \exists N \in \mathbb{N}\ \forall n \geq N : \ x_n = 0\}$. Consider the linear operator

$$T\colon c_c \subset c_0 \to \ell^1, \qquad (Tx)_n = n x_{n+1}.$$

(a) Is T extendable to a bounded linear operator $T\colon c_0 \to \ell^1$? Justify your answer.

(b) Compute the adjoint of T, namely determine

$$T^*\colon D_{T^*} \subset (\ell^1)^* \to (c_0)^*.$$

Notice that the characterization of the subspace D_{T*} is also required.

(c) Prove that the operator T is closable. Define the domain $D_{\overline{T}}$ of its closure and determine an element belonging to the set $D_{\overline{T}} \setminus c_c$.

Problem 5. **[10 points]**

Let H be a Hilbert space over $\mathbb{R}$ and let $A\colon H \to H$ be linear, compact and self-adjoint.

(a) State the spectral theorem for A.

Now, suppose the existence of two complementary and mutually orthogonal subspaces $H', H'' \subset H$ that are A-invariant, meaning that

$$H = H' \oplus^\perp H'', \qquad A(H') \subset H', \qquad A(H'') \subset H''.$$

(b) Show that each of the restricted operators $A' := A_{|H'}$ and $A'' := A_{|H''}$ is also compact and self-adjoint.

Assume now that A is non-negative definite (i. e. $(Ax, x) \geq 0$ for all $x \in H$).

(c) State the Courant–Fischer characterization of the eigenvalues of A.

(d) Denoted by $\lambda_1, \lambda_1', \lambda_1''$ the first (namely: the *largest*) eigenvalue of A, A', A'' respectively, show that

$$\lambda_1 = \max\{\lambda_1', \lambda_1''\}.$$

Problem 6. **[10 points]**

(a) State the Arzelà–Ascoli theorem.

Consider $S \subset C^0([0,1])$ a closed linear subspace (where $C^0([0,1])$ is endowed with its standard sup norm). Suppose that the following implication holds:

$$f \in S \;\Rightarrow\; f \in C^1([0,1]).$$

(b) Show that the operator $\mathrm{id}\colon S \to C^1$ given by $\mathrm{id}(f) = f$ has closed graph.

(c) Show that S is finite dimensional.

Problem 7. **[10 points]**

Let $\Omega \subset \mathbb{R}^d$ be a measurable set with finite measure.

(a) Show that, for any $1 \le p \le \infty$ and any $N \ge 0$, the set

$$F_{p,N} := \{f \in L^p(\Omega) : \|f\|_{L^p} \le N\}$$

is a closed subset of $L^1(\Omega)$.

(b) In the case $d = 1$ and $\Omega = [0,1]$, let $X \subseteq L^1([0,1])$ be a closed vector subspace and assume that

$$X \subset \bigcup_{1<p\le\infty} L^p([0,1]).$$

Prove that $X \subseteq L^q([0,1])$ for some $1 < q \le \infty$.

(c) In the same setting as in (b) show that there exists a constant $C > 0$ such that

$$\|f\|_{L^q} \le C\|f\|_{L^1} \quad \forall f \in X.$$