Stable Minimal Surfaces in $\mathbb{R}^3$

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November 26, 2019

We follow the pages 29-32 of the lectures by Brian White [3].

During the very first presentation, we looked at a specific variational problem to deduce that a minimal surface would have to pass the first derivative test, which led us to the minimal surface equation. The derivative of the area functional under deformations gave us valuable information about minimal surfaces. However, we only looked at the first derivative, and we can ask ourselves if the study of higher order derivatives is also helpful. The answer to this question is yes, as we shall shortly see by considering the second derivative.

Let $M \subset \mathbb{R}^3$ be an orientable minimal surface, and let $\phi_t : M \to \mathbb{R}^3$ be a smooth one-parameter family of smooth maps.

Definition 1. *We say that M is stable if*

$$\left(\frac{d^2}{dt^2}\right)\bigg|_{t=0} \text{area}(\phi_t M) \geq 0,$$

for all deformations ϕ_t with $\phi_0(x) \equiv x$ and $\phi_t(y) \equiv y$ for $y \in \partial M$.

Remark 2. *The smooth maps ϕ_t defined above induce a one-parameter family of surfaces $M_t = \phi_t(M)$, each of them with boundary ∂M, and such that $M_0 = M$.*

We want to consider normal vector fields

$$X(x) = \left(\frac{d}{dt}\right)\bigg|_{t=0} \phi_t(x),$$

which can be written as

$$X = u\nu,$$

where $u : M \to \mathbb{R}$ is a function and ν is the unit normal vector field (see figure 1). Moreover, note that $\phi_t(x) = x$ for $x \in \partial M$ implies that $u \equiv 0$ on ∂M.

Theorem 3 (The second variation formula)**.** *Under the hypotheses above,*

$$\begin{aligned}\left(\frac{d^2}{dt^2}\right)\bigg|_{t=0} \text{area}(\phi_t M) &= \int_M (|\nabla u|^2 - |A|^2 u^2)\, dS \\ &= \int_M (-\Delta u - |A|^2 u) u \, dS.\end{aligned}$$

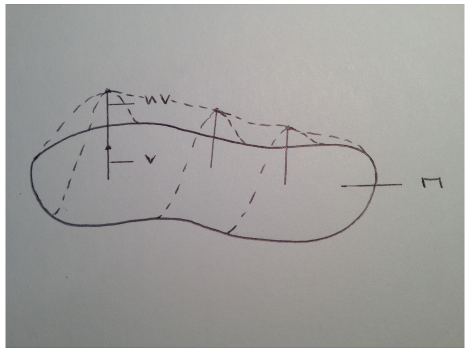

Figure 1: Deformation of the surface M by the function u.

Proof. See pages 38-40 of the reference [2]. □

Using the second variation formula, we may state the following corollary.

Corollary 4. *A surface M as above is stable if and only if for any function $u \in C_c^\infty(M)$ such that $u \equiv 0$ on ∂M, we have*

$$\int_M (|\nabla u|^2 - |A|^2 u^2)\, dS \geq 0.$$

Let us now give the following definitions, some of them already known from previous presentations.

Definition 5. *The total curvature of M is given by*

$$TC(M) = \int_M |K|\, dS,$$

where K is the Gaussian curvature, $K = \kappa_1 \kappa_2$.

Definition 6. *A topological space X is called simply connected if it is path-connected and any loop in X defined by $f : S^1 \to X$ can be contracted to a point.*

Definition 7. *Let $M \subset \mathbb{R}^d$ be a compact, smooth manifold without boundary. Assume that the metric in M is induced by the scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^d$. A curve $\gamma : I \subset \mathbb{R} \to M$ is a geodesic if the covariant derivative (defined by Riemannian connection), $\frac{D}{dt}(\gamma'(t))$, is equal to zero for all $t \in I$.*

Definition 8. *A regular surface M is said to be complete when for every point $p \in M$, any parametrized geodesic $\gamma : [0, \varepsilon) \to M$ of M, starting from $\gamma(0) = p$, may be extended into a parametrized geodesic $\tilde{\gamma} : \mathbb{R} \to M$, defined on the entire line $\mathbb{R}$.*

Definition 9. *Let $p, q \in M$ be points on the surface M. We define the distance function $d_M(p, q) : M \times M \to [0, \infty)$ as*

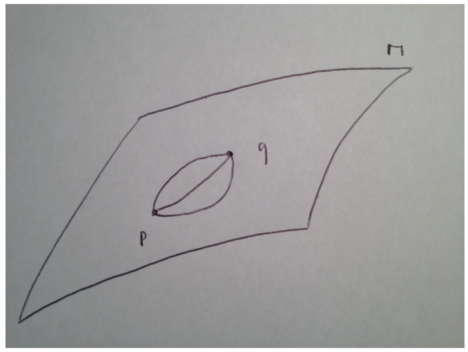

Figure 2: To determine $d_M(p,q)$, we take the infimum of the length of all curves connecting p and q.

$$d_M(p,q) := \textit{inf}\,\left\{\textit{length}(\gamma) : \gamma \textit{ is a curve in } M \textit{ connecting } p \textit{ and } q\right\},$$

see figure 2.

Remark 10. *As a distance function, $d_M(\cdot,\cdot)$ defines a metric on M.*

Definition 11. *The geodesic ball $B_r(p)$ centered at the point $p \in M$ with radius r is the union of all points $x \in M$ such that $d_M(p,x) < r$, where $d_M(p,q)$ is the length of a minimizing geodesic between p and q in M (see figure 3).*

We now give some results which will be useful later.

Lemma 12. *If we define $r_p(x) := d_M(p,x)$, then its gradient $\nabla_M r_p$ exists almost everywhere.*

Proof. Although this is a general fact about distance functions, let us give a formal proof. We first show that r_p is Lipschitz continuous. Indeed, using the reverse triangular inequality, we may write that for all $x_1, x_2 \in M$,

$$|d_M(p,x_1) - d_M(p,x_2)| \leq d_M(x_1,x_2) \leq |x_1 - x_2|,$$

where in the last term, $|\cdot|$ denotes the three-dimensional Euclidean norm, and thus the last equality holds as $d_M(x_1,x_2)$ is always greater or equal than the length of the direct line connecting x_1 and x_2. Thus r_p is indeed Lipschitz continuous with Lispschitz constant 1. Let us now have a look at the following theorem.

Theorem 13 (Rademacher's theorem)**.** *If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is Lipschitz continuous, then f is differentiable almost everywhere.*

Since a geodesic ball is open, we may now use this theorem on M to conclude the proof. □

Lemma 14. *We have that $|\nabla_M r_p| = 1$ almost everywhere.*

Proof. Once again, this is a general fact about distance functions, best understood analysing the norm function $x \mapsto \|x\|$ restricted to M, similarly to what we did in lemma 12. $\square$

We are now ready to investigate what follows.

Proposition 15. *Let M be a complete, simply connected surface with $K \leq 0$. Let $A(r) = A_p(r)$ be the area of the geodesic ball $B_r(p)$ of radius r about some point $p \in M$. Let*

$$\theta(M) = \lim_{r \to \infty} \frac{A(r)}{\pi r^2}.$$

Then

$$\theta(M) = 1 - \frac{1}{2\pi} \int_M K\, dS = 1 + \frac{TC(M)}{2\pi}.$$

Remark 16. *Note that $\theta(M)$ is an intrinsic analog of $\Theta(M)$, the density at infinity of a properly immersed minimal surface (without boundary) in Euclidean space.*

Proof of Prop. 15. Let $L(r)$ be the length of ∂B_r. From a previous presentation, we know the following theorem to be true.

Theorem 17 (Coarea formula)**.** *Let M be a submanifold of $\mathbb{R}^n$, and $h : M \to \mathbb{R}$ be a proper (i.e. $h^{-1}((-\infty, r])$ is compact for all $r \in \mathbb{R}$) Lipschitz function on M. Then for every locally integrable function f on M and $r \in \mathbb{R}$ we have that*

$$\int_{h \leq r} f |\nabla_M h| = \int_{-\infty}^{r} \int_{h=\tau} f\, d\tau,$$

where $\nabla_M h$ is the tangential projection of the gradient.

Define $f = 1$, $h(x) = r_p(x)$. Using theorems 12 and 14, we find that

$$\int_{B_r(p)} |\nabla_M r_p|\, d\mu = \int_0^r \left(\int_{\partial B_\tau(p)} d\omega \right) d\tau = \int_0^r |\partial B_\tau(p)|\, d\tau = \int_0^r L(\tau)\, d\tau,$$

where $d\mu$ denotes the volume measure on M and $d\omega$ denotes the volume measure on $\partial B_\tau(r)$. But we also have that

$$\int_{B_r(p)} |\nabla_M r_p|\, d\mu = \int_{B_r(p)} 1\, d\mu = |B_r(p)| = A(r),$$

and thus in this case $A' = L$. So

$$A'' = L' = \int_{\partial B_r} k\, ds = 2\pi - \int_{B_r} K\, dS.$$

The formula for L' follows from the first variation of arc length, where k is the geodesic curvature of ∂M, and the last equality then follows from the Gauss-Bonnet theorem given below.

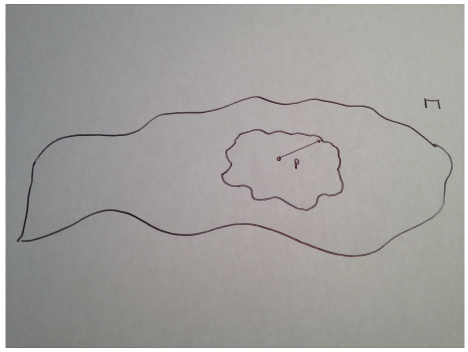

Figure 3: The geodesic ball in M centered at the point p

Theorem 18 (Gauss-Bonnet)**.** *Suppose M is a compact two-dimensional Riemannian manifold with boundary ∂M. Let K be the Gaussian curvature of M, and let k be the geodesic curvature of ∂M. Then*

$$\int_M K\,dS + \int_{\partial M} k\,ds = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

Remark 19. *In our case, $\chi(M) = 1$. Indeed, under the above hypotheses, we may reduce our problem to the case of an open disk, and the Euler characteristic of an open disc is equal to 1 because the disc is contractible.*

Thus

$$\lim_{r\to\infty} A''(r) = 2\pi - \int_M K\,dS = 2\pi + TC(M).$$

The results then follows by L'Hôpital's rule. □

Corollary 20. *If M as above is a minimal surface in $\mathbb{R}^3$ and if $\theta(M) < 3$, then M is plane.*

Proof. Let us recall the following theorem already used in another presentation.

Theorem 21. *If $M \subset \mathbb{R}^3$ is a complete, orientable minimal surface of total curvature $< 4\pi$, then M is a plane.*

Now if $\theta(M) < 3$, then by proposition 15 we have that $TC(M) < 4\pi$, and therefore M is a plane. □

Lemma 22 (Pogorelov)**.** *Let $M \subset \mathbb{R}^3$ be a simply connected, minimal surface. Suppose $B_R(p)$ is a geodesic ball in M of radius R about some point $p \in M$ such that the interior of B_R contains no points of ∂M, i.e. such that $\operatorname{dist}(p, \partial M) \geq R$. If $A(R) := \operatorname{area}(B_R) > \frac{4}{3}\pi R^2$, then B_R is unstable.*

Proof. We may assume that $M = B_R$. To prove instability, it suffices by corollary 4 to find a function u in B_R with $u \equiv 0$ on ∂B_R such that $Q(u) < 0$, where

$$Q(u) = \int_M (|\nabla u|^2 - |A|^2 u^2)\, dS = \int_M |\nabla u|^2\, dS + 2\int_M K u^2\, dS. \tag{1}$$

The second equality holds because

$$4|H|^2 = |A|^2 + 2K$$

for any surface. Moreover, we used the fact that the mean curvature H vanishes at all points for any minimal surface. Let r and θ be geodesic polar coordinates in M centered at the point p. Thus the metric has the form

$$ds^2 = dr^2 + g^2\, d\theta^2$$

for some nonnegative function $g(r, \theta)$ such that

$$g(0,0) = 0, \quad g_r(0,0) = 1.$$

The Gauss curvature can be computed to be equal to

$$K = -\frac{g_{rr}}{g}.$$

Remark 23. *The above results about geodesic polar coordinates are not trivial and should be computed and proved.*

Thus the second integral in equation (1) becomes

$$\begin{aligned} Q_2(u) &:= 2\int_M u^2 K\, dS = 2\int_0^{2\pi}\int_0^R u^2 K g\, dr\, d\theta \\ &= -2\int_0^{2\pi}\int_0^R u^2 g_{rr}\, dr\, d\theta. \end{aligned}$$

Integrating by parts twice and inserting

$$u(R,\theta) = 0, \quad u(0,\theta) = u(0,0), \quad g_r(0,\theta) = 1, \quad g(0,\theta) = 0,$$

gives

$$\begin{aligned} Q_2(u) &= -2\int_0^{2\pi} \Bigg(u(R,\theta)^2 g_r(R,\theta) - u(0,\theta)^2 g_r(0,\theta) \\ &\qquad - \left((u^2)_r(R,\theta) g(R,\theta) - (u^2)_r(0,\theta) g(0,\theta) - \int_0^R (u^2)_{rr} g\, dr \right) \Bigg)\, d\theta \\ &= 4\pi u(0)^2 - 2\int_0^{2\pi}\int_0^R (u^2)_{rr} g\, dr\, d\theta \\ &= 4\pi u(0)^2 - 4\int_0^{2\pi} (u_r)^2 g\, dr\, d\theta - 4\int_0^{2\pi}\int_0^R u u_{rr} g\, dr\, d\theta \\ &= 4\pi u(0)^2 - 4\int_M (u_r)^2\, dS - 4\int_0^{2\pi}\int_0^R u u_{rr} g\, dr\, d\theta. \end{aligned} \tag{2}$$

Now let

$$u(r,\theta) = u(r) = \frac{R-r}{R},$$

so that $u(r)$ decreases linearly from $u(0) = 1$ to $u(R) = 0$. Then the last integral in (2) vanishes, and

$$(u_r)^2 = |\nabla u|^2 = \frac{1}{R^2},$$

so combining (1) and (2) gives

$$Q(u) = 4\pi - \frac{3}{R^2}A(R),$$

which is negative if $A(R) > \frac{4}{3}\pi R^2$. □

Remark 24. *Note that the above function u is not smooth at the origin. However, this unique problematic point does not affect the integral, as one can show using an iterative approximation argument.*

Let us now have a look at the following results.

Theorem 25 (Fischer-Colbrie/Schoen)**.** *Suppose M is an oriented minimal hypersurface in $\mathbb{R}^n$. Then M is stable if and only if there is a positive solution of*

$$\Delta u + |A|^2 u = 0$$

on $M \setminus \partial M$.

Proof. This is a general fact about the lowest eigenvalue of self-adjoint, second-order elliptic operators. □

Corollary 26. *Let M be as in theorem 25. If M is stable, then so is its universal cover.*

Proof. Lift the function u from M to its universal cover. □

We now formulate the most important result of this presentation.

Theorem 27.

1. *A complete, stable, orientable minimal surface in $\mathbb{R}^3$ must be a plane.*

2. *If M is a stable, orientable minimal surface in $\mathbb{R}^3$, then*

$$|A(p)|\,\mathrm{dist}(p, \partial M) \leq C$$

for some $C < \infty$.

Remark 28. *The two above statements are equivalent.*

Proof of Thm. 27. Using corollary 26, we may assume that M is simply connected. Suppose by contradiction that M is not a plane. Then by corollary 20,

$$\theta(M) \geq 3 > \frac{4}{3},$$

so

$$\frac{A(r)}{\pi r^2} > \frac{4}{3}$$

for large r. But then M is unstable by lemma 22, a contradiction to the assumptions. □

References

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[2] Tobias H Colding and William P Minicozzi. *A course in minimal surfaces*. eng. Vol. vol. 121. Graduate studies in mathematics. Providence, R.I: American Mathematical Society, 2011. ISBN: 978-0-8218-5323-8.

[3] Brian White. *Lectures on Minimal Surface Theory*. 2013. arXiv: 1308.3325 [math.DG].