Density at Infinity and Isoperimetric Inequality

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October 30, 2019

Every picture in this paper comes from the beautiful *“ Minimal Surface Archive”* by Matthias Weber, which is an archive of more than 150 minimal surfaces with descriptions and high-quality images. I strongly recommend to everyone interested in minimal surfaces to take a look at their page: `http://www.indiana.edu/~minimal/archive/`.

Density at Infinity

Definition (Density at infinity). *Let M be an m-dimensional immersed minimal surface without boundary in $\mathbb{R}^n$. For any fixed $p \in \mathbb{R}^n$ we call $\Theta(M,p) = \lim_{r\to\infty} \Theta(M,p,r)$ the* ***density of M at infinity*** *at p, indeed the limit exists. Furthermore, $\Theta(M,p)$ does not depend on the choice of $p \in \mathbb{R}^n$, so we can denote it by $\Theta(M)$.*

By the Monotonicity Theorem we know $\Theta(M,p,r)$ is, for every $p \in \mathbb{R}^n$, an increasing function of r over $r \in (0,+\infty)$. Thus $\lim_{r\to\infty} \Theta(M,p,r)$ exists in $[1,+\infty]$. Note that we allow the limit to be infinite. Moreover, we also have to prove it does not depend on the choice of p. Indeed, for every $p, q \in \mathbb{R}^n$ note

$$B_r(p) \subseteq B_{r+|p-q|}(q).$$

Then, by the definition of density ratio, for every $r \in (0,+\infty)$ we have

$$\begin{aligned}\Theta(M,p,r) &= \frac{\mu(M \cap B_r(p))}{\mu(B_r(0) \subset \mathbb{R}^m)} \leq \frac{\mu(M \cap B_{r+|p-q|}(q))}{\mu(B_r(0) \subset \mathbb{R}^m)} \\ &= \frac{\mu(M \cap B_{r+|p-q|}(q))}{\mu(B_{r+|p-q|}(0) \subset \mathbb{R}^m)} \cdot \frac{\mu(B_{r+|p-q|}(0) \subset \mathbb{R}^m)}{\mu(B_r(0) \subset \mathbb{R}^m)} \\ &= \Theta(M,q,r+|p-q|) \cdot \frac{(r+|p-q|)^m}{r^m},\end{aligned}$$

taking the limit as $r \to \infty$ gives $\Theta(M,p) \leq \Theta(M,q)$. Clearly, swapping p and q also gives the reverse inequality with the same argument, thus $\Theta(M,p) = \Theta(M,q)$ for every $p, q \in \mathbb{R}^n$. That is $\Theta(M,p)$ does not depend on the choice of p and so we can denote it just by $\Theta(M)$. Thus, when calculating density at infinity, we can choose the point $p \in \mathbb{R}^n$ center of the ball that makes computations easier.

Note that, since M is locally Euclidean at every $p \in \mathbb{R}^n$, by the Monotonicity Theorem for every $r > 0$ we have

$$\Theta(M) \geq \Theta(M,p,r) \geq \lim_{r\to 0^+} \Theta(M,p,r) \geq 1.$$

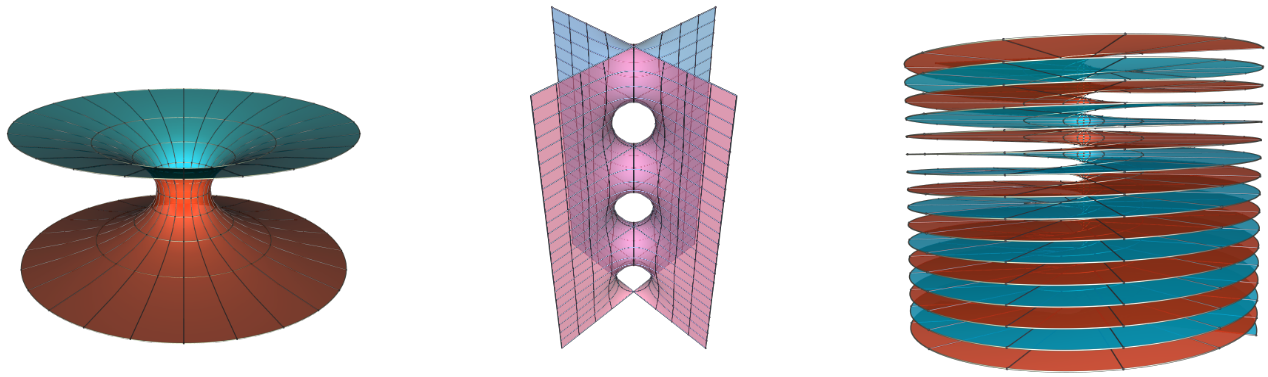
In fact, as long as M is smooth, $\lim_{r\to 0^+} \Theta(M,p,r)$ is a nonnegative integer equal to the multiplicity of M at p. Indeed, it is always equal to one if M is embedded[1] in $\mathbb{R}^n$, but it can be greather than one if M is not embedded.

Example 1. *The density at infinity of a plane is 1. Indeed, take an m-dimensional plane immersed in $\mathbb{R}^n$ and choose $p \in \mathbb{R}^n$ being on the plane. Then $\Theta(M) = \Theta(M,p) = \lim_{r\to\infty} \Theta(M,p,r) = 1$ since $\Theta(M,p,r) \equiv 1$ as a function of the radius r. Almost in the same way one can show that the density at infinity of $k \geq 1$ planes is just k.*

Example 2. *The catenoid and the Scherk's surface are two nontrivial surfaces with density at infinity equal to 2. To see this fact take for example the catenoid and choose a ball centered in its center. Now, in order to find its density at infinity, instead of fixing the surface and let the radius of the ball tend to infinity, we fix the ball and shrink the catenoid by a factor $1/r$ for $r \geq 0$. Letting $r \to \infty$ we would see the catenoid converging to two horizontal parallel discs, hence the density at infinity of the catenoid is 2. The same argument works for Scherk's surface, see Figure* 1(a) *and Figure* 1(b) *below.*

[1] recall that M is embedded in $\mathbb{R}^n$ if there exists an immersion $f : M \to \mathbb{R}^n$ such that M is homeomorphic to $f(M)$, where $f(M)$ inherits the subspace topology of $\mathbb{R}^n$. In particular, if M is embedded in $\mathbb{R}^n$, then $f(M)$ can not have self-intersections.

Example 3. *The helicoid is clearly an example of surface with density at infinity equals to infinity, see Figure* 1(c).



(a) Catenoid ($\Theta(M) = 2$) (b) Scherk's surface ($\Theta(M) = 2$) (c) Helicoid ($\Theta(M) = +\infty$)

Figure 1: Examples of surfaces with density at infinity $\Theta(M) \neq 1$.

Isoperimetric Inequality

We state and prove, only for minimal surfaces, one formulation of the Isoperimetric inequality for manifolds.

Theorem 1 (Isoperimetric inequality)**.** *Let M be a smooth, compact m-dimensional submanifold in $\mathbb{R}^n$. Let $|M|$ be the m-dimensional measure of M and $|\partial M|$ be the $(m-1)$-dimensional measure of $|\partial M|$. Then*

$$|M| \leq C_m \left(|\partial M| + \int_M |\vec{H}| \, dS \right)^{\frac{m}{m-1}},$$

*for some constant $C_m > 0$ **only** dependent on m.*

For what concerns the proof, we will focus on the case $m = 2$ and we will also require M to be minimal surface. The proof is not too hard once we have established some preliminary results.

Theorem 2. *Let M be a compact m-dimensional minimal submanifold of $\mathbb{R}^n$. Then*

$$m|M| = \int_{\partial M} x \cdot \nu \, ds \,,$$

where $x = (x_1, x_2, \ldots, x_n)$.

Proof. The proof is a straightforward application of the Generalized Divergence Theorem, with vector field $X(x) = x$. Since M is a minimal submanifold, then $\vec{H} = 0$. Furthermore by definition of divergence on a manifold we have

$$\operatorname{div}_M(X) = \sum_{i=1}^{n} e_i \cdot \partial_{e_i}(X) = \sum_{i=1}^{n} e_i \cdot e_i = m \,,$$

combining everything we get the desired result. □

Surprisingly, the following inequality is the keystone of the presented proof of the isoperimetric inequality, in our particular setting. However, other inequalities of the same form, namely Poincaré-Type inequalities, works for more general settings. These inequalities, provided that the function has null average, allow estimating L^p-norm with the L^p-norm of the gradient.

Theorem 3 (Wirtinger's inequality)**.** *Let $f \in C^1(\mathbb{R},\mathbb{R})$ be a 2π periodic function such that*

$$\int_0^{2\pi} f(t)\,dt = 0\,.$$

Then

$$\int_0^{2\pi} f^2(t)\,dt \le \int_0^{2\pi} (f'(t))^2\,dt.$$

Proof. The hypothesis allows us to expand $f(t)$ in Fourier series at every point $t \in [0,2\pi]$, thus we have

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right),$$

moreover $a_0 = 0$ since f has vanishing average on $[0,2\pi]$ by hypothesis. Now the conclusion follows from Parseval's identity and the fact that we can expand also $f'(t)$ in Fourier series on $[0,2\pi]$, indeed

$$\int_0^{2\pi} f^2(t)\,dt = \sum_{n=1}^{\infty}(a_n^2+b_n^2) \le \sum_{n=1}^{\infty} n^2(a_n^2+b_n^2) = \int_0^{2\pi} (f'(t))^2\,dt.$$

□

Corollary 1. *Let T be positive real number, and let $f \in C^1(\mathbb{R},\mathbb{R})$ be a T periodic function such that*

$$\int_0^{T} f(t)\,dt = 0\,.$$

Then

$$\int_0^{T} f^2(t)\,dt \le \frac{T^2}{4\pi^2}\int_0^{T} (f'(t))^2\,dt.$$

Proof. The result follows by Theorem 3 applied to the function $\tilde{f}(x) = f\left(\frac{Tx}{2\pi}\right)$, which has period 2π. □

Remark. *This is a simple form of Poincaré's inequality for regular functions in one dimension.*

Now we are ready to prove the isoperimetric inequality with some additional assumptions.

Theorem 4 (Baby isoperimetric inequality)**.** *Let M be a smooth, compact 2-dimensional minimal submanifold in $\mathbb{R}^3$. Let $|M|$ be the 2-dimensional measure of M and $|\partial M|$ be the 1-dimensional measure of $|\partial M|$. Suppose also that ∂M is connected. Then*

$$|M| \le \frac{1}{4\pi}|\partial M|^2.$$

Proof. In force of Theorem 2 with $m = 2$ we have

$$2|M| = \int_{\partial M} x\cdot\nu\,ds\,.$$

Since $\|\nu\| = 1$, by Cauchy–Schwarz inequality

$$\begin{aligned} 2|M| = \int_{\partial M} x\cdot\nu\,ds \le \int_{\partial M}\|x\|\,ds \le \left(\int_{\partial M} ds\right)^{\frac{1}{2}}\left(\int_{\partial M}\|x\|^2\,ds\right)^{\frac{1}{2}} \\ = |\partial M|^{\frac{1}{2}}\left(\int_{\partial M}(x_1^2+x_2^2+x_3^2)\,ds\right)^{\frac{1}{2}}. \end{aligned} \quad (1)$$

Now, without loss of generality assume the centroid of ∂M to be the origin $(0,0,0) \in \mathbb{R}^3$. This can be done by moving M in the ambient space.

Since all the cordinates of the centroid are zero, we have

$$\int_{\partial M} x_1\, ds = \int_{\partial M} x_2\, ds = \int_{\partial M} x_3\, ds = 0\,.$$

This assumption may seem almost random at this point, but this will be the key property in order to apply Corollary 1 of Wirtinger's inequality, as it requires functions to have null average. Let $|\partial M| = \ell$ for simplicity, and consider a parametrization $\varphi : [\,0,\ell\,] \to \mathbb{R}^3$ of ∂M by arc lenght, that is in particular $\|\dot{\varphi}(t)\| = 1$ for every $t \in [\,0,\ell\,]$. Thus for every $i \in \{1,2,3\}$ we have $\varphi_i(0) = \varphi_i(\ell)$ and $\dot{\varphi}_i(0) = \dot{\varphi}_i(\ell)$. Moreover, since

$$\int_{\partial M} x_i\, ds = \int_0^\ell \varphi_i \|\dot{\varphi}\|\, ds = \int_0^\ell \varphi_i\, ds = 0\,,$$

all the hypothesis of Corollary 1 are satisfied by $\varphi_i(\cdot)$ for every $i \in \{1,2,3\}$, we have

$$\int_{\partial M} x_i^2\, ds = \int_0^\ell \varphi_i^2 \|\dot{\varphi}\|\, dt = \int_0^\ell \varphi_i^2\, dt \le \frac{\ell^2}{4\pi^2} \int_0^\ell \dot{\varphi}_i^2\,,$$

summing up this estimate for $i \in \{1,2,3\}$ gives

$$\int_{\partial M} (x_1^2 + x_2^2 + x_3^2)\, ds \le \frac{\ell^2}{4\pi^2} \int_0^\ell \|\dot{\varphi}\|^2\, ds = \frac{l^3}{4\pi^2} = \frac{|\partial M|^3}{4\pi^2}.$$

Equation (1) and the above estimate allows us to conclude that

$$|M| \le \frac{1}{2}\, |\partial M|^{\frac{1}{2}} \left(\int_{\partial M} (x_1^2 + x_2^2 + x_3^2)\, ds \right)^{\frac{1}{2}} \le \frac{1}{2}\, |\partial M|^{\frac{1}{2}} \cdot \frac{|\partial M|^{\frac{3}{2}}}{2\pi} = \frac{|\partial M|^2}{4\pi}.$$

□

Remark. *The constant we found is **sharp**, since equality holds for circles immersed in* $\mathbb{R}^3$*. One can find equality cases by starting from those of Cauchy-Schwarz inequality. Another important remark is that there is nothing special about* $n = 3$ *in the presented proof, indeed exactly the same argument works for a 2-dimensional submanifold of* $\mathbb{R}^n$ *with the same assumptions, for all* $n \ge 3$.

References

[1] Osserman, Robert. *A survey of minimal surfaces.* Second edition. Dover Publications, Inc., New York, 1986. vi+207 pp.

[2] Colding, Tobias H., and William P. Minicozzi. *A course in minimal surfaces,* Vol. 121. American Mathematical Soc., 2011.

[3] Matthias Weber, *Minimal Surface Archive*: `http://www.indiana.edu/~minimal/archive/`