First Variation of Area

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Introduction

By the end of the lecture we will have introduced a necessary condition for a surface with boundary to have the least area among the surfaces with the same boundary. This will also provide us with a new interpretation of the mean curvature H.

Notation

Let us start by explaining the setting our goal will be achieved in and by fixing some notation.

Let $U \subset \mathbb{R}^2$ be an open set, $\varphi : U \to \mathbb{R}^3$ be a parametrization of the surface $\varphi(U) = \Sigma$, i.e. a smooth function such that

$$\left\{ \frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_2}(u) \right\} = \{\varphi_{u_1}(u), \varphi_{u_2}(u)\} \quad \text{are linearly independent } \forall u \in U.$$

The function φ is also called an immersion. Notice that we will not require φ to be bijective. This will be useful in giving a parametrization of the catenoid in the final example. Since its atlas is made of a single chart, such surface is called elementary.

Furthermore let us denote with

- $I_u : T_u(U) \times T_u(U) \to \mathbb{R}$, $u \in U$ defined by $I_u(X,Y) = \delta(d\varphi[u](X), d\varphi[u](Y))$, where $\delta(\cdot,\cdot)$ is the inner product of $\mathbb{R}^3$ and $d\varphi[u]$ is the manifold derivative of φ at the point u, also often called the differential of φ. $I(\cdot,\cdot)$ is called first fundamental form. Notice that it is bilinear, symmetric and positively defined because of the properties of the inner product $\delta(\cdot,\cdot)$.

- g_{ij}, $1 \leq i,j \leq 2$ will denote an element of the representation matrix of $I(\cdot,\cdot)$. $g_{ij}(u) = I(\partial u_i, \partial u_j) = \delta(\varphi_{u_1}(u), \varphi_{u_2}(u))$. Although both the first fundamental form and g_{ij} depend on u, for simplicity of notation we will omit it. Furthermore, $g_{11} = E$, $g_{12} = g_{21} = F$ and $g_{22} = G$.

- $\nu : U \to \mathbb{R}^3$ will denote the unit normal vector to Σ defined by $\nu(u) = \frac{\varphi_{u_1}(u) \wedge \varphi_{u_2}(u)}{\|\varphi_{u_1}(u) \wedge \varphi_{u_2}(u)\|}$.

- $II : T_u(U) \times T_u(U) \to \mathbb{R}$ defined by $II(X,Y) = \delta(d\nu(X), d\varphi(Y))$ will denote the second fundamental form.

- h_{ij}, $1 \leq i,j \leq 2$ will denote an element of the representation matrix of $II(\cdot,\cdot)$. $h_{ij}(u) = II(\partial u_i, \partial u_j) = \delta(\nu_{u_i}(u), \varphi_{u_j}(u))$. Notice that since $\delta(\nu(u), \varphi_{u_i}(u)) = 0$, $\forall u \in i = 1,2$, then differentating this expression by Scharz's Theorem we get:

$$\delta(\nu_{u_i}(u), \varphi_{u_j}(u)) = -\delta(\nu(u), \varphi_{u_i u_j}(u)) = \delta(\nu_{u_j}(u), \varphi_{u_i}(u)). \tag{1}$$

  Hence the second fundamental form is symmmetric. We will denote, $h_{11} = e$, $h_{12} = h_{21} = f$ and $h_{22} = g$.

- The mean curvature H is the trace of the representation matrix of $d\nu$. It can also be expressed in terms of the elements of the representation matrix of I and II by:

$$H = \frac{Eg + Ge - 2Ff}{EG - F^2} = \frac{Eg + Ge - 2Ff}{\det(\{g_{ij}\})}.$$

Minimizing area

We will now use a standard argument in calculus of variations to provide a necessary condition for the problem of finding the surface that minimizes area given a boundary.

Let $\Delta \subset U$ be a bounded open set. $\varphi(\partial\Delta)$ is the boundary of the minimizing problem.

Let $l \in C_c^\infty(\Delta, \mathbb{R})$ and $\lambda \in \mathbb{R}$. $\tilde{\varphi} : U \to \mathbb{R}^3$ be defined by

$$\tilde{\varphi}(u) = \varphi(u) + \lambda l(u)\nu(u).$$

Notice that, although smooth, $\tilde{\varphi}$ may not be a surface. We also have that $\tilde{\varphi}(\partial\Delta) = \varphi(\partial\Delta)$.

Let us now differentiate $\tilde{\varphi}$:

$$\tilde{\varphi}_{u_i}(u) = \varphi_{u_i}(u) + \lambda(l_{u_i}(u)\nu(u) + l(u)\nu_{u_i}(u)).$$

We will from now on omit the dependence from $u \in U$ for the sake of simplicity of notation.

Let us define $\tilde{g}_{ij} : U \to \mathbb{R}$, $1 \leq i, j \leq 2$ as

$$\begin{aligned}\tilde{g}_{ij} &= \delta(\tilde{\varphi}_{u_i}, \tilde{\varphi}_{u_j}) = \delta(\varphi_{u_i}, \varphi_{u_j}) + \lambda(\delta(\varphi_{u_i}, l_{u_j}\nu + l\nu_{u_j}) + \delta(\varphi_{u_j}, l_{u_i}\nu + l\nu_{u_i})) + \lambda^2 c_{i,j} \\ &= g_{ij} + 2\lambda l h_{ij} + \lambda^2 c_{i,j}.\end{aligned}$$

Where $c_{i,j}$ is a smooth function (because it is obtained by multiplication and composition of smooth functions). We have also used equation (1) and the definitions of ν, g_{ij} and h_{ij} have been used.

Let us now calculate the determinant of the matrix $\{\tilde{g}_{ij}\}$ by explicit computation:

$$\det(\{\tilde{g}_{ij}\}) = \det(\{g_{ij}\}) + 2\lambda l(Eg + Ge - 2Ff) + \lambda^2 c. \tag{2}$$

where $c : U \times \mathbb{R} \to \mathbb{R}$ is a smooth function dependent on u and λ (in particular, it is a second degree polynomial in λ).

Lemma 1. *There exists a constant $\varepsilon > 0$ such that $\tilde{\varphi}(U)$ restricted to Δ is a regular surface for every λ such that $|\lambda| < \varepsilon$.*

Proof. To prove the statement we will show that $\det(\{\tilde{g}_{ij}\}) \neq 0$ for such values of λ for all $u \in \Delta$. Consider equation (2). For $\lambda = 0$ we have that $\det(\{\tilde{g}_{ij}\}) = \det(\{g_{ij}\}) > 0$ since $\{g_{ij}\}$ is positively defined for every $u \in \overline{\Delta}$. Let

$$m(\lambda) = \min_{u \in \overline{\Delta}} \det(\{\tilde{g}_{ij}\}),$$

which is well defined as $\overline{\Delta}$ is compact. Since $m(0) > 0$, if m was continuous there would exist an $\varepsilon > 0$ such that $m(\lambda) > 0$ for every $|\lambda| < \varepsilon$ and the theorem would be proved. □

Given a function $f : A \to \mathbb{R}$ continuous, $A \subset \mathbb{R}^n$, notice that $\inf_{x \in A} f = -\max_{x \in A}(-f)$. Hence, in order to prove the lemma, it suffices to show the following:

Claim 1. *Let $K \in \mathbb{R}^n$ be compact and $f : K \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then*

$$\max_{x \in K}(f(x, \cdot)) : \mathbb{R} \to \mathbb{R}.$$

is continuous.

Proof. The function is well defined since K is compact and $f(\cdot, y)$ is continuous. In particular $f(\cdot, y)$ is uniformly continuous for every $y \in \mathbb{R}$ because of Heine-Kantor's theorem. We need to show that $\forall \varepsilon > 0, y \in \mathbb{R} \quad \exists \delta(\varepsilon, y)$ such that $|\max_{x \in K}(f(x, y)) - \max_{x \in K}(f(x, y_0))| < \varepsilon$ if $|y - y_0| < \delta$.

Let $\varepsilon > 0$ be fixed. By assumption we know that $\exists \tilde{\delta}(\varepsilon, y)$ such that $|f(x, y) - f(x_0, y_0)| < \varepsilon$ if $\|(x, y) - (x_0, y_0)\| < \tilde{\delta}$.Therefore

$$|\max_{x \in K}(f(x, y)) - \max_{x \in K}(f(x, y_0))| \leq \max_{x \in K} |f(x, y) - f(x, y_0)| < \varepsilon \quad \forall y_0 : |y - y_0| \leq \tilde{\delta}.$$

Therefore is suffices to take $\delta = \tilde{\delta}$. □

Lemma 2. *There exist an $M > 0$ and $\tilde{\varepsilon}$ such that the inequality:*

$$\left| \sqrt{\det(\{\tilde{g}_{ij}\})} - \left(\sqrt{\det(\{g_{ij}\})} + \frac{2l(Eg + Ge - 2Ff)}{2\sqrt{\det(\{g_{ij}\})}} \lambda \right) \right| < M\lambda^2. \tag{3}$$

holds for all $u \in \Delta$ and $|\lambda| < \tilde{\varepsilon}$.

Proof. For simplicity of notation, we denote

- $a = \sqrt{\det(\{g_{ij}\})}$;
- $b = 2l(Eg + Ge - 2Ff)$;
- c as in equation (2);

The first term of the inequality above becomes:

$$\left|\sqrt{a + \lambda b + \lambda^2 c} - \left(\sqrt{a} + \frac{b}{2\sqrt{a}}\lambda\right)\right|.$$

By rationalizing we have

$$\left|\frac{a + \lambda b + \lambda^2 c - \left(a + \frac{b^2}{4a}\lambda^2 + b\lambda\right)}{\sqrt{a + \lambda b + \lambda^2 c} + \left(\sqrt{a} + \frac{b}{2\sqrt{a}}\lambda\right)}\right| = \lambda^2 f(u, \lambda).$$

Where

$$f(u, \lambda) = \left|\frac{c - \frac{b^2}{4a}}{\sqrt{a + \lambda b + \lambda^2 c} + \left(\sqrt{a} + \frac{b}{2\sqrt{a}}\lambda\right)}\right|.$$

Let us note the following facts:

- f is continuous when it is defined;
- $f(u, 0) < \infty$ for every $u \in \overline{\Delta}$ because $a = \sqrt{\det(\{g_{ij}\})} > 0$;

Therefore we can apply Claim 1 to show that $\max_{u \in \overline{\Delta}}(f(u, \cdot))$ is continuous. Hence, there exists $\tilde{\varepsilon} > 0$ such that $\max_{u \in \overline{\Delta},\ \lambda \in [-\tilde{\varepsilon}, \tilde{\varepsilon}]}(f(u, \lambda)) = \hat{M} < +\infty$. The proof follows by taking $M = \hat{M} + 1$. □

A sufficient condition

Summarizing, we have proven that $\tilde{\Sigma} = \tilde{\varphi}(\overline{\Delta})$ is a regular surface and that inequality (3) holds for every $\lambda < \min\{\varepsilon, \tilde{\varepsilon}\}$.

Let us know define the function

$$A(\lambda) = \int_{\Delta} \sqrt{\det(\{\tilde{g}_{ij}\})}\, du_1 du_2.$$

We will now show that every critical point of the functional area just defined is a surface with mean curvature identically 0.

Theorem 1.

$$A'(0) = 2\int_{\Delta} H(u) l(u) \sqrt{\det(\{g_{ij}\})}\, du_1 du_2.$$

Proof. By integrating inequality (3) we obtain:

$$\left|A(\lambda) - A(0) - \lambda \int_{\Delta} \frac{2l(Eg + Ge - 2Ff)}{2\sqrt{\det(\{g_{ij}\})}}\right| < M|\Delta|\lambda^2.$$

Dividing by λ we get

$$\left|\frac{A(\lambda) - A(0)}{\lambda} - \int_{\Delta} \frac{l(Eg + Ge - 2Ff)}{\sqrt{\det(\{g_{ij}\})}}\right| < M|\Delta|\lambda.$$

By letting λ tend to zero we have the claim. □

Notice that by choosing $l(u) = \frac{1}{2}$ we have that the following formula holds:

$$A'(0) = \int_{\Delta} H(u) \sqrt{\det(\{g_{ij}\})}\, du_1 du_2.$$

Corollary 1. *If a regular surface with boundary Σ has the least area out of all the other regular surfaces with the same boundary, then its mean curvature is identically zero.*

Proof. Suppose not. Then without loss of generality there exists a point $\tilde{u} \in U$ such that $H(\tilde{u}) = k > 0$. Then by continuity there exists $r > 0$ such that $\overline{B_r(\tilde{u})} \subset \Delta$ and $H(u) \geq \frac{H(\tilde{u})}{2}$ for all $u \in B_{\frac{r}{2}}(\tilde{u})$. Choose l to be the cutoff function which is equal to 1 on $B_{\frac{r}{2}}(\tilde{u})$ and 0 in $\Delta \setminus \overline{B_r(\tilde{u})}$. We have then that

$$A'(0) > \frac{\pi r^2}{8} H(\tilde{u}) > 0.$$

Therefore, by noticing that with such choice of l, $\tilde{\varphi}(\Delta) = \Sigma_\lambda$ is a regular surface for every $|\lambda| < \min\{\varepsilon, \tilde{\varepsilon}\}$ with the same boundary as Σ, by assumption $A(\lambda) \geq A(0)$ would imply $A'(0) = 0$, contradiction. ☐

Example 1. *Let us verify that the two surfaces mentioned in the mind experiment the first lecture, namely:*

- *Catenoid:*

$$\varphi(u_1, u_2) = (c \cosh \frac{u_2}{c} \cos u_1, c \cosh \frac{u_2}{c} \sin u_1, u_1) \qquad u_1, u_2 \in \mathbb{R}.$$

- *Plane:*

$$\varphi(u_1, u_2) = (u_1, u_2, 0).$$

have mean curvature 0.