Minimal Surfaces Seminar: The Concentration Theorem

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These notes follow Brian White's notes on minimal surfaces [Whi16] on pages 27 to 29. The main topic will be the concentration theorem 3, but first, we look at another equivalence between a global theorem and a local curvature estimate. The following global theorem is a consequence of the monotonicity theorem:

Theorem 1. *If $M \subseteq \mathbb{R}^n$ is a proper minimal smooth submanifold without boundary and if $\Theta(M) \leq 1$, then M is a plane.*

Proof. Take a point p in M. Then, by the monotonicity theorem and since $\partial M = \emptyset$, we have for all $r > 0$

$$1 \leq \Theta(M, p, r) \leq \Theta(M).$$

Thus, our assumption forces $\Theta(M) = 1$. By the monotonicity theorem again, this implies that M intersects $\partial B(p, r)$ orthogonally for every r. Geometrically, this means that M is invariant under dilation about the point p, which exactly means that M is a cone with vertex p. However, M is also assumed to be smooth, and thus M must be a union of planes (possibly with multiplicity) passing through p. Finally, since we have established that $\Theta(M) = 1$, M is in fact a single plane with multiplicity 1. □

Now, we state an equivalent theorem, whose statement is local. This is a special case of a more general result called Allard's regularity theorem.

Theorem 2. *There exist $\lambda > 1$, $\epsilon > 0$, and $C < \infty$ with the following property. If $M \subseteq \mathbb{R}^n$ is minimal,* $\operatorname{dist}(p, \partial M) \geq R$ *and* $\Theta(M, p, R) \leq \lambda$*, then*

$$\sup_{q \in B(p,\epsilon R)} |\mathrm{A}| \operatorname{dist}(q, \partial B(p, \epsilon R)) \leq C,$$

where $p \in \mathbb{R}^n$, dist *is the Euclidean distance and* A *denotes the second fundamental form.*

This can be proved using theorem 1 and proceeding as in the proof of the 4π curvature estimate theorem. We will not spell out the details here.

Moreover, if M is as in the hypotheses of theorem 1, then $\operatorname{dist}(p, \partial M) \geq R$ and $\Theta(M, p, R) \leq \lambda$ hold for every R and thus the second fundamental must be zero. We can then deduce that M is a single plane with multiplicity 1 as we did in the proof of theorem 1. This shows that theorem 2 implies theorem 1.

Now, we move onto the main topic of this talk, namely the concentration theorem. This theorem deals with sequences of minimal surfaces with total curvatures that are uniformly bounded, but not bounded by some $\lambda < 4\pi$. In this case, we get smooth subsequential convergence except at a finite number of points, which are those points where the curvature is concentrated. More precisely:

Theorem 3 (Concentration theorem)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be an open subset. Suppose that $(M_k)_{k \in \mathbb{N}} \subset \Omega$ are two-dimensional, oriented, and embedded minimal surfaces. Assume that $\partial M_k \subset \partial\Omega$ and that there exists $\Lambda < \infty$ such that the total curvatures* $\mathrm{TC}(M_k) \leq \Lambda$ *for all $k \in \mathbb{N}$. Moreover, assume that the areas of the M_k's are uniformly bounded in compact subsets of Ω. Then, after passing to a subsequence there exists a set $S \subset \Omega$ of at most $\frac{\Lambda}{4\pi}$ points such that the M_k's converge locally and smoothly in $\Omega \setminus S$ to a limit minimal surface M. Furthermore the surface $M \cup S$ is a smooth and embedded minimal surface.*

Remark 1. The concentration also holds when the ambient space is $\mathbb{R}^n$ for $n > 4$ if we replace 4π with 2π at every step. This is because in higher dimension, Osserman's theorem and the 4π curvature estimate hold for 2π instead of 4π

Example 1. To illustrate what theorem 3 says, take M_k to be the catenoid dilated by a factor $1/k$. As k grows larger, this converges to a plane with multiplicity 2, and the convergence is smooth everywhere but at the origin, where the curvature blows up.

In order to prove the concentration theorem, we will need a few standard concepts from functional analysis that were taken from [Str13] (in German).

Definition 1. A metric space X is called *separable* if it has a countable dense subset.

We will need this concept in the specific case where $X = C_c^0(\mathbb{R}^n)$ is the space of compactly supported continuous functions on $\mathbb{R}^n$. It is a fact that this space is separable. This can be proven using Weierstrass approximations.

Definition 2. Let X be a normed vector space, and let X^* denote the space of all continuous linear maps from X to $\mathbb{R}$. We say that a sequence (f_k) in X^* *w^*-converges* to $f \in X^*$ if for every $x \in X$, we have $f_k(x) \to f(x)$ in $\mathbb{R}$.

The theorem that we will use in the proof of theorem 3 is the so-called Banach-Alaoglu theorem:

Theorem 4 (Banach-Alaoglu)**.** *Let X be a normed separable space, and let $(f_k) \subset X^*$ be a bounded sequence of linear operators. Then there exists a subsequence of f_k that w^*-converges to some $f \in X^*$.*

We will not prove the theorem here as it is not really the topic of these notes, but this is a really standard result in functional analysis.

The idea of the proof of theorem 3 is to use the 4π curvature estimate and the basic compactness theorem to get the convergence that we want. But these theorems can only be used if the uniform bound on the total curvatures is smaller than 4π. As we have seen with the examples of the catenoids above, the problematic points are those where the curvature blows up. The strategy will be to construct a sequence of measures that put the mass where the curvature is big and to use them to find those problematic points. We need one additional concept, which we do not define in full generality.

Definition 3. A sequence of measures μ_k on $\mathbb{R}^n$ converges weakly to a measure μ if for every compactly supported continuous function f, we have

$$\int f d\mu_k \to \int f d\mu.$$

We are now ready for the proof of theorem 3.

Proof. We start by defining a sequence of measures on Ω: set $\mu_k(U) = \mathrm{TC}(M_k \cap U)$ for any Borel set $U \subset \mathbb{R}^3$. Let us define the associated linear operators T_k on the space of compactly supported continuous functions $C_c^0(\mathbb{R}^3)$ as follows: $T_k(f) = \int f d\mu_k$. Those linear operators are bounded in the operator norm, since the μ_k's are bounded and thus we can use Banach-Alaoglu to get a w^*-converging subsequence, which by our construction corresponds to a weakly converging subsequence of measures. In short, by passing to a subsequence, we can assume that the μ_k's converge weakly to a limit measure μ. Moreover, it holds by construction that $\mu(\Omega) \leq \Lambda$.
Let S be the set of point p such that $\mu(\{p\}) \geq 4\pi$. Then the finiteness of μ implies that $|S| \leq \frac{\Lambda}{4\pi}$. Suppose that $x \in \Omega \setminus S$. Then, $\mu(\{x\}) < \lambda < 4\pi$. Moreover, but outer regularity of μ, there exists a closed ball $B = B(x, r) \subset \Omega$ with $\mu(B) < \lambda$. This implies that $\mathrm{TC}(M_k \cap B) = \mu_i(B) < \lambda$ for k large enough. Using the 4π curvature estimate (theorem 23 in [Whi16]), it follows that the second fundamental form $|\mathrm{A}_k|$ is uniformly bounded on $B(x, r/2)$. Since we now have a local uniform bound on the second fundamental form, we can use the basic compactness theorem (theorem 22 in [Whi16]) to get subsequential local smooth convergence to a minimal surface M on $\Omega \setminus S$.
We now show that $M \cup S$ is a smooth surface. Take $p \in S$. By translation we can assume that $p = 0$ without loss of generality. Note that there exists $\epsilon > 0$ such that $\mu(B(0, \epsilon) \setminus \{0\})$ is arbitrarily small. If we dilate the surface M about zero by a sequence of numbers tending to infinity, using the 4π curvature estimate and the basic compactness theorem again, we can find a subsequence that converges smoothly on $\mathbb{R}^3 \setminus \{0\}$ to a limit minimal surface with total curvature zero, which can only be a union of planes. Since we assumed the M_i's to have uniformly bounded areas on compact subsets, the number of those planes must be finite. Indeed, since we are dilating about 0, we have that $\Theta(M) = \Theta(0, r, M)$, which is finite by our assumption. This construction shows that, for r small enough, the surface $M \cap (B(0, r) \setminus \{0\})$ is topologically a finite union of punctured discs. In fact, it is possible to show that the smooth subsequential convergence of the dilated surfaces to planes implies that $M \cap (B(0, r) \setminus \{0\})$ is even conformally a finite union of punctured discs. Let $F : D \setminus \{0\} \to \mathbb{R}^3$ be a conformal parametrization for one of those punctured discs. We know from a previous talk that F is also harmonic. Using tools from complex analysis, it is possible to show that isolated singularities of bounded harmonic functions are removable, and thus

F extends smoothly to the whole disc D. This shows that $M \cup S$ is a smooth surface by the maximum principle for minimal surfaces (see exercise sheet 4).
Finally by the basic compactness theorem in $\Omega \setminus S$ and our previous analysis around S, if we assume that $M \subset \mathbb{R}^3$ is not embedded (possibly with multiplicity), then portions of M must intersect each others transversely. But then the smooth convergence $M_k \to M$ away from S implies that some of the M_k's also have self-intersections and thus are not be embedded. This concludes the proof. □

It the surfaces M_k are assumed to be simply connected, we get a stronger statement:

Theorem 5. *Let us assume, in the setting of the concentration theorem 3, that the surfaces $M_k \subset \Omega \subset \mathbb{R}^3$ are simply connected. Then $S = \emptyset$.*

Proof. For a contradiction, let us assume that S contains a point p. By the concentration theorem, the surface $M \cup \{p\}$ near p is a smooth embedded surface. Thus we can take a small ball B around p such that $\partial B \cap M$ is very nearly circular, and $B \cap S = \{p\}$. Then, for k large enough the smooth convergence of the M_k's to M away from S imply that $M_k \cap \partial B$ is the union of Q almost circular curves, for some multiplicity Q. This holds because the M_k's are embedded, as the pathological cases of perturbation of circles transversed multiple times are excluded.

It can then be shown that $M_k \cap B$ is the union of simply connected components. Since every such component has a nearly circular boundary for k large enough, its total curvature is very small by the Gauss-Bonnet theorem. But then the curvatures of the M_k's are uniformly bounded and very close to zero on compact subsets of the interior of B by the 4π curvature estimate, which is a contradiction for subsets containing p, where the total curvature is bigger than 4π. □

The concentration theorem is only useful if it is possible to obtain uniform bounds on the total curvature of the M_k's. Fortunately, this is possible in many situations. For example, suppose that the $M_k \subset \mathbb{R}^3$ all have the same finite topological type. (For the readers familiar with algebraic topology, a topological space X is of finite type if all homology groups $H_n(X)$ are finitely generated.) In particular, this implies that all the M_k's have the same Euler characteristic. Furthermore, suppose that we also have reasonable boundary curves ∂M_k, or more precisely:

$$\sup_k \int_{\partial M_k} |\kappa_{\partial M_k}| \, ds < \infty,$$

where $\kappa_{\partial M_k}$ denotes the curvature vector of the curve ∂M_k. Then, we have that $\sup_k \mathrm{TC}(M_k) < \infty$ by the Gauss-Bonnet theorem.

To conclude, we present a theorem that makes all the necessary assumptions of the concentration theorem hold.

Theorem 6. *Let Ω be an open subset of $\mathbb{R}^n$. Let (M_k) be a family of minimal surfaces in Ω with $\partial M_k \subset \partial\Omega$. Suppose that*

$$\sup_k \operatorname{genus}(M_k) < \infty.$$

Furthermore assume that

$$\sup_k \operatorname{area}(M_k \cap U) < \infty, \quad \text{for } U \subset\subset \Omega.$$

Then we have that

$$\sup_k \mathrm{TC}(M_k \cap U) < \infty, \quad \text{for } U \subset\subset \Omega,$$

where $U \subset\subset \Omega$ means $U \subset \Omega$ and $\overline{U}$ compact in Ω.

References

[Whi16] B. White, *Introduction to minimal surface theory.* Geometric analysis, 387-438, IAS/ Park City Math. Ser., 22. American Mathematical Society, Providence, RI, 2016.

[Str13] M. Struwe, *Funktionalanalysis I und II.* Lecture notes, ETH Zrich, 2013/14.