Seminar on Minimal Surfaces
Choi-Schoen theorem, part I

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Monday 9th December, 2019

We will begin by recalling few definitions:

Definition 1. Let M be a manifold, Σ an embedded k-manifold and (Σ_n) be a sequence of embedded k-manifolds. We say that (Σ_n) *converges smoothly to* Σ if for each point $p \in \Sigma$ there exists a neighbourhood U of p in M and an integer N such that for each $n \geq N$ we have that $\Sigma_n \cap U$ is a graph of a smooth function u_n over $\Sigma \cap U$ and $u_n \to 0$ in the $\mathcal{C}^k$-norm for each k.

Definition 2. Let (M, g) be a Riemannian manifold and let ∇ denote the Levi-Civita connection. Then the tensor

$$R(X,Y)Z \coloneqq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is called the *Riemann curvature tensor*. Let u, w be tangent vectors to M at p. We let

$$\mathrm{Ric}(u,w) \coloneqq \mathrm{trace}(v \mapsto R(u,v)w)$$

and call Ric the *Ricci curvature tensor.*

Definition 3. We say that a manifold is *closed* if it is compact and has no boundary.

The aim of this talk is to prepare the proof of the following theorem:

Theorem 4 (Choi-Schoen)**.** *Let M be a smooth closed three dimensional manifold with positive Ricci curvature. The space of closed embedded surfaces of genus g in M is compact in the smooth topology.*

To prove the theorem we will need other preliminary results.

Theorem 5 (Choi-Wang)**.** *If M is a simply connected closed manifold with $\mathrm{Ric}_M\, k$ for some $k \in \mathbb{R}$ and $\Sigma \subseteq M$ is a closed embedded minimal hypersurface, then the first Dirichlet eigenvalue of Δ_Σ is at least $k/2$.*

To prove theorem 5 we need the following formula which we take as given.

Lemma 6 (Reilly)**.** *Let u be a smooth function on a bounded domain Ω. Then*

$$\int_\Omega \left(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - (\Delta u)^2\right) = \int_{\partial\Omega} \left(A((\nabla u)^T, (\nabla u)^T) - 2u_n \Delta_{\partial\Omega} u + H u_n^2\right)$$

where u_n is the normal derivative of u and H is the mean curvature of $\partial\Omega$.

Remark 7. In the proof of theorem 5 we use the following important fact: If Σ is a closed hypersurface in a simply connected manifold M then $M \setminus \Sigma$ consists of two connected components M_1, M_2 and $\partial M_1 = \partial M_2 = \Sigma$.

Proof of theorem 5. We have that, by remark 7, that $\Sigma = \partial\Omega$ for some bounded domain Ω. Let λ be a Dirichlet eigenvalue of $\Delta_{\partial\Omega}$ and f a corresponding eigenfunction. Let u be the solution to the Dirichlet problem

$$\begin{cases} \Delta_\Omega u = 0 \\ u|_{\partial\Omega} = f \end{cases} .$$

Now we are going to apply lemma 6 to the function u. We begin by estimating the left hand side in the lemma:

$$\int_\Omega \left(|\nabla^2 u|^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) \geq \int_\Omega \operatorname{Ric}(\nabla u, \nabla u) \geq k \int_\Omega |\nabla u|^2.$$

Integration by parts gives

$$\int_{\partial\Omega} u_n u = \int_{\partial\Omega} u\langle \nabla u, n\rangle = \int_\Omega u\Delta u + \int_\Omega \langle \nabla u, \nabla u\rangle = \int_\Omega |\nabla u|^2$$

so the right hand side becomes:

$$\int_{\partial\Omega} \left(A((\nabla u)^T, (\nabla u)^T) - 2u_n \Delta_{\partial\Omega} u + H u_n^2\right) = \int_{\partial\Omega} A((\nabla u)^T, (\nabla u)^T) + 2\lambda \int_\Omega |\nabla u|^2.$$

Therefore, the lemma gives:

$$2\lambda \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} A((\nabla u)^T, (\nabla u)^T) \geq k \int_\Omega |\nabla u|^2. \tag{1}$$

If we do exactly the same computations with $\Omega' := M \setminus \overline{\Omega}$ in place of Ω we will end up with the inequaltiy

$$2\lambda \int_{\Omega'} |\nabla \tilde{u}|^2 - \int_{\partial\Omega} A((\nabla \tilde{u})^T, (\nabla \tilde{u})^T) \geq k \int_{\Omega'} |\nabla \tilde{u}|^2 \tag{2}$$

where $\tilde{u}$ is defined analogously to u, on Ω'. Note that $(\nabla u)^T = (\nabla u)^T = \nabla_\Sigma f$ so adding (1) and (2) we get

$$2\lambda \left(\int_\Omega |\nabla u|^2 + \int_{\Omega'} |\nabla \tilde{u}|^2\right) \geq k \left(\int_\Omega |\nabla u|^2 + \int_{\Omega'} |\nabla \tilde{u}|^2\right)$$

so $\lambda \geq k/2$. □

For proving theorem 4 we will need an a priori upper bound on the area and the total curvature of an embedded minimal surfaces of a fixed genus.

Corollary 8. *Let M be a three dimensional Riemannian manifold with* $\operatorname{Ric}_M \geq k > 0$ *and Σ be a two dimensional closed embedded minimal surface of M of genus g. Then*

$$\operatorname{Area}(\Sigma) \leq \frac{16\pi(1+g)}{k} \tag{3}$$

and, if the norm of the sectional curvature of M is bounded by K, i.e. $|K_M| \leq K$, then

$$\int_\Sigma |A|^2 \leq \frac{32\pi K(1+g)}{k} + 8\pi(g-1). \tag{4}$$

Proof. To prove (3) we use a theorem of Yang and Yau which says that if Σ is an embedded two dimensional surface of genus g and λ is the first Dirichlet eigenvalue of Δ_Σ then

$$\lambda \cdot \mathrm{Area}(\Sigma) \leq 8\pi(1+g).$$

From this inequality and theorem 5 the inequality (3) is clear.

Let e_1, e_2 be an orthonormal basis for the tangent space of Σ at some point. Then, by Gauss' equation, we have

$$\begin{aligned} K_\Sigma = K_\Sigma(e_1,e_2) &= K_M(e_1,e_2) + \langle A(e_1,e_1), A(e_2,e_2)\rangle - \langle A(e_1,e_2), A(e_1,e_2)\rangle \\ &= K_M(e_1,e_2) - \frac{1}{2}|A|^2 \leq K - \frac{1}{2}|A|^2 \end{aligned} \tag{5}$$

where in the second last step we used that, by minimality, we have

$$\langle A(e_1,e_1), A(e_2,e_2)\rangle = -\langle A(e_1,e_1), A(e_1,e_1)\rangle = -\langle A(e_2,e_2), A(e_2,e_2)\rangle$$

By Gauss-Bonnet we have

$$\int_\Sigma K_\Sigma = 2\pi\chi(M) = 4\pi(1-g). \tag{6}$$

Combining (5), (6) and inequality (3) we get

$$\int_\Sigma |A|^2 \leq 2K\,\mathrm{Area}(\Sigma) + 8\pi(g-1) \leq \frac{32\pi K(1+g)}{k} + 8\pi(g-1).$$

□

To state our last proposition we shall recall the following definition:

Definition 9. Let (M,d) be a metric space and X, $Y \subseteq M$ be two nonempty compact sets. For a set $A \subseteq M$ we set

$$A^\delta := \{x \in M \mid d(x,A) < \delta\}.$$

Then we set

$$d_H(X,Y) := \inf\{\delta \mid Y \subseteq X^\delta \text{ and } X \subseteq Y^\delta\}$$

and call $d_H(X,Y)$ the *Hausdorff distance between* X *and* Y.

Proposition 10. *Let M^3 be a closed three dimensional manifold and (Σ_n) a sequence of closed embedded submanifolds of M of genus g with*

$$\mathrm{Area}(\Sigma_n) \leq A \tag{7}$$

and

$$\int_{\Sigma_n} |A_{\Sigma_n}|^2 \leq B. \tag{8}$$

for some constants A and B. Then there exists a finite set of points $S \subseteq M$ and a subsequence (Σ_{n_k}) such that for every $l \in \mathbb{N}$, Σ_{n_k} converges uniformly on compact sets of $M \setminus S$ in the $\mathcal{C}^l$ topology to a smooth minimal surface $\Sigma \subseteq M$ which has genus at most g and satisfies (7) and (8). Furthermore, the subsequence converges to Σ in Hausdorff distance.

Example 11. Consider S^3. The space $\{\Sigma \subseteq S^3 \mid \Sigma \text{ is isometric to } S^2\}$ is compact in the smooth topology. Hence every sequence of 2-spheres in S^3 has a subsequence which converges to a 2-sphere.

References

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