Minimal Surfaces: Isothermal Parameters

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Often in mathematics we have some degree of freedom to choose certain objects. One could then try to buy some control over other quantities with this freedom. For example in numerical integration, we have $2n$ degrees of freedom to choose n nodes and n weights. Gauss quadrature pays this freedom for a specific choice which leads to a quadrature rule of order $2n$.

Similarly, we have for our surfaces the freedom to choose a parametrization. One way we could use this freedom is to try to find a parametrization that preserves angles, i.e. such that angles in the parameter plane are mapped conformally to angles on the surface. Such parameters are called **isothermal parameters**. Indeed, isothermal parameters do exist locally for all C^2-surfaces. We refer to [Oss86] Lemma 4.4 for a proof for minimal surfaces.

Let $x(u)$, $u = (u_1, u_2)$ be a parametrization of a surface S. Recall that the first fundamental form $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $g(v, w) = \delta(dx_u(v), dx_u(w)) = dx_u(v) \cdot dx_u(w)$ gives an inner product on S at the point u. Let $g = (g_{ij})$, $i, j \in \{1, 2\}$ denote the representation matrix of the first fundamental form g. Since g is an inner product on S, it describes the angles on the surface.

Lemma 1. *In the notation above, u_1, u_2 are isothermal parameters if and only if*

$$g_{ij} = \lambda^2 \delta_{ij} \tag{1}$$

for some $\lambda = \lambda(u) > 0$.

Proof. A parametrization is isothermal if and only if it preserves angles at every point, so let us fix a specific point u. If (g_{ij}) has the form of equation (1), then (g_{ij}) is a scaling and thus preserves angles. Conversely, suppose g preserves angles. Then it holds

$$\frac{g(v, w)}{\|v\|_g \|w\|_g} = \cos\phi = \frac{v \cdot w}{\|v\|_2 \|w\|_2},$$

where $||\cdot||_g$ is the induced norm by g and ϕ is the angle between v and w. Using $g(v, w) = v^T (g_{ij}) w$ and plugging in $v = (1, 0)$, $w = (0, 1)$ we get $g_{12} = g_{21} = 0$. Plugging in $v = (1, -1)$, $w = (1, 1)$ and using positive definiteness of g gives $g_{11} = g_{22}$. Finally (g_{ij}) has to be positive definite, so $g_{11} > 0$. $\square$

Let us fix a point $p = x(a)$ on S. Recall that the mean curvature at p is $H(p) = \frac{1}{2}\mathrm{tr}(h(p))$, where $h(p)$ is the second fundamental form at p. Expressed in terms of derivatives of x we get the formula

$$H(p) = \frac{g_{22} h_{11}(a) + g_{11} h_{22}(a) - 2 g_{12} h_{12}(a)}{2 \det(g_{ij})},$$

where $h_{ij}(a) = \frac{\partial^2 x}{\partial u_1 \partial u_2}(a) \cdot N$ and $N \in T_pS^\perp$ with $\|N\|_2 = 1$. Using isothermal parameters this simplifies to

$$H(p) = \frac{h_{11}(a) + h_{22}(a)}{2\lambda^2}. \tag{2}$$

Since the mixed derivatives vanished, we now have a relation of the Laplacian of x and the mean curvature vector $\overrightarrow{H}$:

Lemma 2. *Let a regular surface S be defined by $x(u) \in C^2$, where u_1, u_2 are isothermal parameters. Then*

$$\Delta x = 2\lambda^2 \overrightarrow{H}$$

where $\overrightarrow{H} = H(p)N$ is the mean curvature vector.

Proof. Since $\overrightarrow{H} \,||\, N \in T_pS^\perp$ we first show that $\Delta x \in T_pS^\perp \perp \text{span}(\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2})$.

From equation (1) we know that $\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_1} = g_{11} = g_{22} = \frac{\partial x}{\partial u_2} \cdot \frac{\partial x}{\partial u_2}$. Differentiating with respect to u_1 and cancelling out the factor 2 gives

$$\frac{\partial^2 x}{\partial u_1^2} \cdot \frac{\partial x}{\partial u_1} = \frac{\partial^2 x}{\partial u_1 \partial u_2} \cdot \frac{\partial x}{\partial u_2}.$$

On the other hand, equation (1) also gives $\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_2} = g_{12} = 0$. Differentiating with respect to u_2 yields

$$\frac{\partial^2 x}{\partial u_1 \partial u_2} \cdot \frac{\partial x}{\partial u_1} + \frac{\partial x}{\partial u_1} \cdot \frac{\partial^2 x}{\partial u_2^2} = 0.$$

Adding up these two equations shows

$$\Delta x \cdot \frac{\partial x}{\partial u_1} = 0,$$

so indeed Δx is perpendicular to $\frac{\partial x}{\partial u_1}$. By symmetry of u_1 and u_2 in equation (1), we can also deduce

$$\Delta x \cdot \frac{\partial x}{\partial u_2} = 0,$$

so $\Delta x \in T_pS^\perp$. Finally, by equation (2) we have

$$\frac{\Delta x}{2\lambda^2} \cdot N = \frac{\frac{\partial^2 x}{\partial u_1^2} \cdot N + \frac{\partial^2 x}{\partial u_2^2} \cdot N}{2\lambda^2} = \frac{h_{11}(a) + h_{22}(a)}{2\lambda^2} = H(p)$$

and hence

$$\frac{\Delta x}{2\lambda^2} = \left| \frac{\Delta x}{2\lambda^2} \right| N = H(p)N = \overrightarrow{H}$$

$\square$

Recall that a minimal surface has by definition vanishing mean curvature everywhere, i.e. its mean curvature vector vanishes identically. It follows immediately the following theorem:

Theorem 1. *Let $x(u) \in C^2(u)$ define a regular surface S in isothermal parameters. Then S is a minimal surface if and only if all coordinate functions $x_k(u)$ are harmonic.*

This gives us another criterion for S being a minimal surface.