FINITE TOTAL CURVATURE AND OSSERMAN'S THEOREM

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The lecture follows [Whi16, pages 15-17].
We begin with a reminder about concepts and definitions which are important for the new theorem presented in this seminar, namely Osserman's Theorem.

Let M be a surface in $\mathbb{R}^3$. The **Gauss map** is a continuous choice of unit normal

$$N : M \to \mathbb{S}^2$$

with $N(p) \in T_pM^\perp$ for every p in M. There are two possible choices of orientation on $\mathbb{S}^2$, and the standard is given by the outward pointing unit normal. The differential

$$d_pN : TM_p \to T_{N(p)}\mathbb{S}^2 \cong T_pM$$

is a self-adjoint linear map and we associate to it the **second fundamental form**, given by

$$A(v, w) = \langle v, dN_p(w)\rangle \quad v, w \in T_pM.$$

In particular, dN is diagonalisable and thus there exists an orthonormal basis $\{E_1, E_2\}$ of T_pM such that $dN_p(E_1) = \lambda_1 E_1, dN_p(E_2) = \lambda_2 E_2$, where λ_1, λ_2 are the **principal curvatures** at p.

Further, we define the **Gaussian curvature** K to be the signed Jacobian of the Gauss map, and the **total (absolute) curvature** is given by

$$TC(M) = \int_M |K| dS = \int_{p\in\mathbb{S}^2} \# N^{-1}(p) dp \tag{1}$$

where the last inequality is justified by the area formula, since $|K| = |\det(dN)|$.
Recall that the **mean curvature**

$$H = \frac{1}{2}\operatorname{tr}(dN) = \frac{\lambda_1 + \lambda_2}{2}$$

vanishes at all points for any minimal surface, which implies that $\lambda_2 = -\lambda_1$. Thus, if M is minimal we have

$$TC(M) = -\int_M K dS.$$

Theorem 1 (Osserman). *Let $M \subset \mathbb{R}^3$ be a complete, connected, orientable minimal surface of finite total curvature*

$$TC(M) = \int_M |K| dS = -\int_M K dS < \infty.$$

Then the following statements hold true.

i) M is conformally equivalent to a compact Riemann surface Σ (one-dimensional complex manifold) minus finitely many points:

$$M \cong \Sigma \backslash \{p_1, \cdots, p_n\}.$$

ii) The Gauss map extends analytically to the punctures.

iii) There exists a non-negative integer m such that for almost every $v \in \mathbb{S}^2$ exactly m points have unit normal $N = v$.

iv) The total curvature of M is equal to $4\pi m$.

v) M is proper in $\mathbb{R}^3$, that is, every sequence which diverges in M also diverges in $\mathbb{R}^3$.

We will not prove the first and the fifth statement. However, the fifth assertion may be proved using results presented in the next seminar. Before continuing, the following observations are essential.

For a minimal surface M, $dN_p : T_pM \to T_pM$ is linear symmetric and angle preserving. With respect to a suitable choice of basis, its operator matrix is given by

$$\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}.$$

By choosing N to be the inward pointing unit normal on $\mathbb{S}^2$ and thus inverting the orientation on the arrival space $T_pM \cong T_{N(p)}\mathbb{S}^2$, the corresponding matrix becomes

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

The determinant $\det(dN_p) = \lambda^2 > 0$ is now positive, making the map orientation preserving. In particular, identifying $\mathbb{S}^2$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and applying the Cauchy-Riemann equations immediately shows that for this choice of orientation, the Gauss map is holomorphic. We require the following theorem from complex analysis before we begin the proof of Osserman's Theorem.

Theorem 2 (Picard's Theorem)**.** *Let M be a Riemann surface, and w a point in M. If $f : M\backslash\{w\} \to \mathbb{S}^2$ is a holomorphic function with essential singularity at w, then on any open subset containing w the function f attains all but at most two points of $\mathbb{S}^2$ infinitely often.*

Proof of Osserman's Theorem. We prove three of the statements.

ii) By (i), we know M is conformally equivalent to a compact Riemann surface Σ minus finitely many points $\{p_1, \cdots, p_n\}$. Let $U \subset \Sigma$ be a neighbourhood of one of the punctures p. If p is a removable singularity, the Gauss map $N : U\backslash\{p\} \to \mathbb{S}^2$ is meromorphic and therefore extends to p. Otherwise, by Picard's theorem, N takes all but two values in $\mathbb{S}^2$ infinitely many times. This would imply that $\int_U |K|dS = \infty$ by Eq.(1), which is a contradiction. Therefore, N must extend continously and analytically to U. Applying this to all poles $\{p_1, \cdots, p_n\}$ proves the result.

iii) The degree of a continuous function between compact oriented manifolds of equal dimension can be thought of as the number of times the domain is ‚wrapped' around the range under the map. Depending on orientation, this number may be positive or negative but it is always an integer. Formally, the degree of the Gauss map is given by

$$\deg(N) = \sum_{p;N(p)=q} \text{sign} \det(dN_p) \tag{2}$$

where q is a regular value. We can interpret this equation in the following way. In a neighbourhood of each regular point, a smooth map is a local diffeomorphism which is either orientation preserving or reversing. We give a positive or a negative sign to each regular point to systemise this information. The sum of all signs is the **mapping degree**. It can be shown that the degree does not depend on the choice of regular value q (see [Mil97, Chapter 5]). By Sard's theorem, the set of all regular values is a dense subset of $\mathbb{S}^2$, while the set of critical values has Hausdorff measure zero when we view $\mathbb{S}^2$ as a subspace of $\mathbb{R}^3$.

Our previous considerations show the Gauss map $\Sigma \to \mathbb{S}^2$ is holomorphic with respect to the orientation on $\mathbb{S}^2$ induced by the inward pointing unit normal. Let m be its mapping degree. It now follows from Eq.(2) and the above discussion that m is our required integer.

iv) Using (iii), we may now immediately conclude that the total curvature is given by $4\pi m$, since

$$TC(M) = \int_{\mathbb{S}^2} \#N^{-1}(\cdot)dS.$$

□

A consequence of Osserman's theorem is the following characterization of the plane.

Corollary 3. *If $M \subset \mathbb{R}^3$ is a complete, orientable minimal surface of total curvature less than 4π, then M is a plane.*

Proof. If the total curvature is less than 4π it must be equal to zero by Osserman's Theorem, and thus by Eq.(1) we have $\det(dN) \equiv 0$. For a minimal surface in $\mathbb{R}^3$, this implies that the principal curvatures $\lambda_1 = -\lambda_2$ are zero at every point p in M. In particular $dN \equiv 0$, and the Gauss map is therefore constant along the surface. □

References

[Mil97] John W. Milnor. *Topology from the differential viewpoint.* Princeton University Press, 1997.

[Whi16] B. White. *Introduction to minimal surface theory.* Geometric analysis, 387-438, IAS/Park City Math. Ser., 22. American Mathematical Society, 2016.