Extended Monotonicity Theorem

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1 Introduction

All the theorems given in this note can be easily extended to the case where M is a minimal submanifold of $\mathbb{R}^n$, but in order to keep all the proofs less complicated, we will do the proofs for a minimal surface in $\mathbb{R}^3$.

1.1 Monotonicity Formula

According to the monotonicity theorem the density ratio

$$\Theta(M, p, r) := \frac{area(M \cap B(p, r))}{\pi r^2}$$

is an increasing function of r for $0 < r < dist(P, \partial M)$, when M is a minimal surface in $\mathbb{R}^3$ and P is an arbitrary point in $\mathbb{R}^3$.

The theorem is false when $r > dist(P, \partial M)$. For instance, if $M \subset B(P, R)$ then the density will be a strictly decreasing function for $r > R$. We want to extend this formula and construct a new function which is increasing for all values of r. In order to do this, we will add some new parts to the surface to make the density increasing for all values.

2 Preliminaries

Let M be an arbitrary surface in $\mathbb{R}^3$. We define the mean curvature vector $\vec{H}$ as follows

$$\vec{H} = -2HN,$$

where H is the mean curvature and N is the unit normal of M.

Lemma 2.1. *Suppose $E_1, E_2 : U \to \mathbb{R}^3$ is a local orthonormal frame for TM on an open set $U \subset M$. Then we have*

$$\vec{H}(p) = (E_1(p)(E_1) + E_2(p)(E_2))^{\perp} \quad \forall p \in U,$$

where by $X(p)(Y)$, we mean the derivative of Y in the direction X at P.

Proof. We have

$$\langle E_i, N\rangle \;=\; 0 \quad\Rightarrow\quad E_i(\langle E_i, N\rangle) = 0 \quad i = 1,2.$$

Then we get

$$E_i(\langle E_i, N\rangle) \;=\; \langle E_i(E_i), N\rangle + \langle E_i, E_i(N)\rangle \;=\; 0 \quad i = 1,2$$

$$\Rightarrow\quad \langle E_i(E_i), N\rangle \;=\; -\langle E_i, E_i(N)\rangle \;=\; -\langle E_i, dN(E_i)\rangle \quad i = 1,2$$

$$\Rightarrow\quad \langle (E_1(E_1) + E_2(E_2)), N\rangle \;=\; -\langle E_1, dN(E_1)\rangle - \langle E_2, dN(E_2)\rangle \;=\; -tr(dN),$$

where dN is the Gauss map and $tr(dN) = 2H$. Thus

$$(E_1(E_1) + E_2(E_2))^{\perp} \;=\; \langle (E_1(E_1) + E_2(E_2)), N\rangle N \;=\; -2HN. \quad \blacksquare$$

Consider a vector field $X : M \to \mathbb{R}^3$. We define divergence of X with respect to M at the point $p \in M$ as follows

$$div_M(X) \;=\; \langle e_1, e_1(X)\rangle + \langle e_2, e_2(X)\rangle,$$

where e_1, e_2 is an orthonormal basis for T_pM.

Theorem 2.1 (Generalized Divergence Theorem)**.** *Let $X : M \to \mathbb{R}^3$ be a vector field on M. Then*

$$\int_M div_M(X) dS \;=\; \int_{\partial M} X \cdot \nu_{\partial M} ds - \int_M X \cdot \vec{H} dS.$$

For proof of the theorem we refer to [1].

Lemma 2.2. *Suppose that M is a surface in $\mathbb{R}^3$. Let M_r be the portion of M in the ball $\boldsymbol{B}_r(0)$ and Γ_r be the intersection of M and $\partial \boldsymbol{B}_r(0)$. Define $A(r) = area(M_r)$ and $L(r) = length(\Gamma_r)$. Then we have*

$$A'(r) \geq L(r)$$

with equality if and only if $x \in T_xM \;\forall x \in M \cap \partial \boldsymbol{B}_r(0)$.

Sketch of the Proof. Apply the coarea formula to the Lipschitz function $u(x) = \|x\|$ on M.

3 Extended Monotonicity

Theorem 3.1 (Extended Monotonicity). *Suppose that $M \subset \mathbb{R}^3$ is a compact, minimal surface with boundary Γ, and that $p \in \mathbb{R}^3 \backslash \Gamma$. Let $E = E(p, \Gamma)$ denote the* ***exterior cone*** *with vertex p over Γ :*

$$E = \cup_{q \in \Gamma} \{p + t(q - p) \,|\, t \geq 1\}$$

Let $\tilde{M} = M \cup E$. Then the density ratio

$$\Theta(\tilde{M}, p, r) = \frac{area(\tilde{M} \cap B(p, r))}{\pi r^2}$$

is an increasing function of r for all $r > 0$. Indeed,

$$\left(\frac{d}{dr}\right) \Theta(\tilde{M}, p, r) \geq 0,$$

with equality if and only if (i) $\nu_{\partial M} + \nu_{\partial E} \equiv 0$ on $\Gamma \cap B(p, r)$ and (ii) $\tilde{M}$ intersects $\partial B(p, r)$ orthogonally.

Remark. In the definition of $\Theta(\tilde{M}, p, r)$, we count area with multiplicity. For example, if exactly two portions of E overlap in a region, we count the area of that region twice.

In proving the extended monotonicity, we can assume without loss of generality $p = 0$. As in the proof of monotonicity formula, we will apply the generalized divergence theorem to the vector field $X(x) = x$.

Lemma 3.1. *Let E be the exterior cone over Γ with vertex 0. Among all unit vectors $\boldsymbol{v}$ that are normal to Γ at $x \in \Gamma$, the maximum value of $x \cdot \boldsymbol{v}$ is attained by $\boldsymbol{v} = -\nu_{\partial E}$.*
Consequently, $x \cdot \boldsymbol{v} \leq x \cdot -\nu_{\partial E}$ and therefore $x \cdot (\boldsymbol{v} + \nu_{\partial E}) \leq 0$ for every such vector $\boldsymbol{v}$.

Proof of Theorem 3.1. Let $M_r, E_r, \tilde{M}_r, \Gamma_r$ be the portions of $M, E, \tilde{M}, \Gamma$ inside the ball $\mathbf{B}_r = \mathbf{B}(0, r)$. By the generalized divergence theorem,

$$\int_{M_r} div_M(X) dS = \int_{\partial M_r} X \cdot \nu_{\partial M_r} ds - \int_{M_r} X \cdot \vec{H} dS = \int_{\partial M_r} X \cdot \nu_{\partial M_r} ds \quad (1)$$

since $\vec{H} \equiv 0$ on M. Similarly,

$$\int_{E_r} div_E(X) dS = \int_{\partial E_r} X \cdot \nu_{\partial E_r} ds - \int_{E_r} X \cdot \vec{H} dS = \int_{\partial E_r} X \cdot \nu_{\partial E_r} ds \quad (2)$$

because $H \cdot X \equiv 0$ on E, since X is tangent to E. Also, $div_M X \equiv div_E X \equiv 2$, so the left sides of these equations are $2\, area(M_r)$ and $2\, area(E_r)$. Adding equations (1) and (2) gives

$$2\, area(\tilde{M}_r) = \int_{\partial M_r} X \cdot \nu_{\partial M_r} ds + \int_{\partial E_r} X \cdot \nu_{\partial E_r} ds. \quad (3)$$

Note that ∂M_r consists of two parts: $M \cap \partial \mathbf{B}_r$ and Γ_r. Likewise ∂E_r consists of $E \cap \partial \mathbf{B}_r$ and Γ_r. By combining the two integrals over $M \cap \partial \mathbf{B}_r$ and $E \cap \partial \mathbf{B}_r$, and also combining the two integrals over Γ_r, we can rewrite (3) as

$$2\, area(\tilde{M}_r) \;=\; \int_{\partial \tilde{M}_r} x \cdot \nu_{\partial \tilde{M}_r} ds + \int_{\Gamma_r} x \cdot (\nu_{\partial M_r} + \nu_{\partial E_r}) ds, \qquad (4)$$

where $\partial \tilde{M}_r \;=\; \tilde{M}_r \cap \partial \mathbf{B}_r$.

By lemma 3.1, the second integral is everywhere nonpositive. Thus

$$2\, area(\tilde{M}_r) \leq \int_{\partial \tilde{M}_r} x \cdot \nu_{\partial \tilde{M}_r} ds \leq r.length(\partial \tilde{M}_r) \qquad (5)$$

since $\|x\| = r$ on $\partial \tilde{M}_r$.

Define $A(r) \;=\; area(\tilde{M}_r)$ and $L(r) \;=\; length(\partial \tilde{M}_r)$. By lemma 2.2 and (5), we get

$$\begin{aligned} 2\, A(r) &\leq r.L(r) \leq r.A'(r) \\ \Rightarrow A'(r) - \frac{2}{r}.A(r) &\geq 0 \Rightarrow \frac{A'(r)}{r^2} - \frac{2}{r^3}.A(r) \geq 0 \\ \Rightarrow \left(\frac{A(r)}{r^2}\right)' &\geq 0 \end{aligned}$$

In order to have equality in the previous equation, (5) must be an equality and we should have $A'(r) = L(r)$. The first one gives us the fact that $\nu_{\partial M_r} + \nu_{\partial E_r} \equiv 0$ on Γ_r and the second one gives us the fact that $\tilde{M}$ intersects $\partial \mathbf{B}_r$ orthogonally. ■

Corollary 3.1.1. *Let $p \in M, \Gamma$ and $E = E(p, \Gamma)$ be as in the extended monotonicity theorem. If $p \in M \backslash \Gamma$, then $1 \leq \Theta(M \cup E)$, with equality if and only if $M \cup E$ is a plane, i.e. if and only if M is a star-shaped region (w.r.t p) in a plane.*

Proof. If $p \in int(M)$, we know that

$$\lim_{r \to 0} \Theta(\tilde{M}, p, r) \geq 1 \qquad (6)$$

Then due to the fact that $\Theta(\tilde{M}, p, r)$ is an increasing function of r we get the inequality.
Now, assume that we have $\Theta(M \cup E) = 1$. As before, without loss of generality we can assume that $p = 0 \in int(M)$. Combining this with (6), we get

$$\left(\frac{d}{dr}\right) \Theta(\tilde{M}, 0, r) \;=\; 0 \quad \forall r > 0. \qquad (7)$$

According to theorem 3.1 and (7) we get $\nu_{\partial M} + \nu_{\partial E} \equiv 0$ on Γ and $\tilde{M}$ intersects $\partial \mathbf{B}_r$ orthogonally for all values of r. Therefore

$$x \in T_x M \quad \forall x \in M. \tag{8}$$

According to (8), $X(x) = x$ is tangent to M. We claim that X is outward pointing on the boundary. Indeed, we have

$$X \cdot \nu_{\partial M} = -x \cdot \nu_{\partial E} \geq 0 \quad \forall x \in \Gamma.$$

Let ϕ be the flow of X on M. Due to the fact that M is compact and X is outward pointing, there exists a $\delta > 0$ such that ϕ is defined on $(-2\delta, 0] \times M$. Now we get

$$\begin{aligned} \phi(t,x) &= e^t x \\ \Rightarrow \text{the segment} \quad & [e^{-\delta} x, x] \subset M \\ \Rightarrow \{tx |\, 0 \leq & t \leq 1\} \subset M. \end{aligned}$$

Thus M is star shaped w.r.t the origin and we have

$$\left(\frac{d}{dt}\right)\Big|_{t=0} (tx) \; = \; x \in T_0 M,$$

so M is contained in the plane $T_0 M$. ■

References

[1] Brian C. White. “Lectures on Minimal Surface Theory”. In: 2013.