ETH Zürich, Mathematics department, HS19

Notes for the Seminar on Minimal Surfaces

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Bernstein Problem

Definition 1. *Let $\Omega \subset \mathbb{R}^2$ be an open, regular domain. For a C^2 function $u : \overline{\Omega} \to \mathbb{R}$ we define the $Graph_u$ of u as $Graph_u := \{(x, u(x)) : x \in \Omega\} \subset \mathbb{R}^3$ and we say $Graph_u$ is entire if $\Omega = \mathbb{R}^2$.*

In what follows, we need the following lemma for a bound for the total curvature of a minimal graph.

Lemma 1 (Lemma 1.19 in [CM11])**.** *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, then for all nonnegative Lipschitz function η with $supp(\eta) \subset \Omega \times \mathbb{R}$ we have that*

$$\int_{Graph_u} |A|^2 \eta^2 \leq C \int_{Graph_u} \left|\nabla_{Graph_u} \eta\right|^2 \tag{1}$$

for some constant C.

So it is possible to bound the total curvature $\int_{\text{Graph}_u} |A|^2$ in terms of the energy of a cutoff function.
Goal. If $u : \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, then u is a degree-1 polynomial.
Question. Does Lemma 1 imply $\int_{\text{Graph}_u} |A|^2 = 0$?
We want to emphasize that the following calculations (in Heuristic 1 and Heuristic 2) are purely formal, because the functions we define do not necessarily have sufficient regularity.

Heuristic 1. Assume for instance we work over $\mathbb{R}^2$ instead over a general graph. Then we can find a sequence of cutoffs η_N such that $\eta_N \overset{N\to\infty}{\longrightarrow} 1$ but has energy going to 0. For this purpose fix an integer N and define $\eta_N : \mathbb{R}^3 \to \mathbb{R}$ by

$$\eta_N(x) := \begin{cases} 1 & \text{if } r \leq e^N, \\ 2 - \frac{\log(r)}{N} & \text{if } e^N < r \leq e^{2N}, \\ 0 & \text{if } e^{2N} < r, \end{cases}$$

where $r := |x|$. Since $|\nabla \eta_N| = \frac{1}{Nr}$, the energy of η_N is

$$\int_{\mathbb{R}^2} |\nabla \eta_N|^2 \,\mathrm{d}x \,\mathrm{d}y = 2\pi \int_{e^N}^{e^{2N}} \frac{1}{N^2 r^2} r \,\mathrm{d}r = \frac{2\pi}{N} \overset{N\to\infty}{\longrightarrow} 0,$$

which is what we wanted. This is known as the logarithmic cutoff trick.

Heuristic 2. Suppose now we have a manifold $\Sigma \subseteq \mathbb{R}^3$. For points $x, y \in \Sigma$ we define the intrinsic distance $\mathrm{d}_\Sigma(x,y)$ as the infimum of the lengths of all paths in Σ from x to y. Moreover for $x \in \Sigma$ and $r > 0$ we define the intrinsic ball $B_\Sigma(x,r) \subseteq \Sigma$ as $B_\Sigma(x,r) := \{y \in \Sigma : \mathrm{d}_\Sigma(x,y) < r\}$. Fix $x_0 \in \Sigma$, an integer N and define $\eta_N : \Sigma \to \mathbb{R}$ by

$$\eta_N(x) := \begin{cases} 1 & \text{if } r \leq e^N, \\ 2 - \frac{\log(r)}{N} & \text{if } e^N < r \leq e^{2N}, \\ 0 & \text{if } e^{2N} < r, \end{cases}$$

where $r := \mathrm{d}_\Sigma(x, x_0)$. If the manifold Σ satisfy

$$\mathrm{Vol}\left(\partial B_\Sigma(x_0, r)\right) \leq Cr \tag{2}$$

for some constant C, then by the coarea formula, we have that

$$\int_\Sigma |\nabla_\Sigma \eta_N|^2 \,\mathrm{d}\sigma = \int_0^\infty \int_{\partial B_\Sigma(x_0,r)} |\nabla_\Sigma \eta_N|^2 \,\mathrm{d}\tilde{\sigma}\,\mathrm{d}r = \int_0^\infty |\nabla_\Sigma \eta_N|^2 \,\mathrm{Vol}(\partial B_\Sigma(x_0,r))\,\mathrm{d}r,$$

where $\mathrm{d}\tilde{\sigma}$ denotes the measure on $\partial B_\Sigma(x_0, r)$ and the last equality holds, because $|\nabla_\Sigma \eta_N|^2$ is constant on $\partial B_\Sigma(x_0, r)$. Therefore we conclude that

$$\int_\Sigma |\nabla_\Sigma \eta_N|^2 \,\mathrm{d}\sigma \overset{(2)}{\leq} C \int_0^\infty |\nabla_\Sigma \eta_N|^2 \, r \,\mathrm{d}r.$$

But in general the condition (2) may not be true. However, the argument can be pushed through when Σ satisfy

$$\mathrm{Vol}\left(B_\Sigma(x_0, r)\right) \leq Cr^2 \tag{3}$$

for some constant C. Namely, writing $B_\Sigma(r) := B_\Sigma(x_0, r)$, we have that

$$\begin{aligned}
\int_\Sigma |\nabla_\Sigma \eta_N|^2 \,\mathrm{d}\sigma &\overset{(\heartsuit)}{=} \sum_{l=N+1}^{2N} \int\limits_{B_\Sigma(e^l)\setminus B_\Sigma(e^{l-1})} |\nabla_\Sigma \eta_N|^2 \,\mathrm{d}\sigma \\
&\overset{(\diamondsuit)}{\leq} \sum_{l=N+1}^{2N} \frac{1}{N^2 e^{2l-2}} \int\limits_{B_\Sigma(e^l)\setminus B_\Sigma(e^{l-1})} \mathrm{d}\sigma \\
&= \sum_{l=N+1}^{2N} \frac{1}{N^2 e^{2l-2}} \mathrm{Vol}(B_\Sigma(e^l) \setminus B_\Sigma(e^{l-1})) \\
&\leq \sum_{l=N+1}^{2N} \frac{1}{N^2 e^{2l-2}} \mathrm{Vol}(B_\Sigma(e^l)) \\
&\overset{(3)}{\leq} \sum_{l=N+1}^{2N} \frac{1}{N^2 e^{2l-2}} C e^{2l} = \sum_{l=N+1}^{2N} \frac{Ce^2}{N^2} = \frac{Ce^2}{N} \overset{N\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where $(\heartsuit)$ uses that $\nabla_\Sigma \eta_N$ is supported between $B_\Sigma(e^N)$ and $B_\Sigma(e^{2N})$ and $(\diamondsuit)$ uses that $\frac{1}{Nr_1} \leq \frac{1}{Nr_2}$ if $r_2 \leq r_1$ together with $|\nabla_\Sigma \eta_N| = \frac{1}{Nr}$.
We next combine Lemma 1 and the logarithmic cutoff trick to get a total curvature bound for a minimal graph.

Corollary 1 (Corollary 1.20 in [CM11])**.** *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, $k > 1$ and $B_{kR} \subset \Omega \times \mathbb{R}$ for some $R > 0$, then*

$$\int_{B_{\sqrt{k}R} \cap Graph_u} |A|^2 \leq \frac{C}{\log(k)} \tag{4}$$

for some constant C.

Proof. Set $\Sigma := \mathrm{Graph}_u$. Define cutoff function $\eta : \mathbb{R}^3 \to \mathbb{R}$ by

$$\eta(x) := \begin{cases} 1 & \text{if } r \leq \sqrt{k}R, \\ 2 - \frac{2\log(\frac{r}{R})}{\log(k)} & \text{if } \sqrt{k}R < r \leq kR, \\ 0 & \text{if } kR < r, \end{cases}$$

where $r := |x|$. We have that

$$|\nabla_\Sigma \eta| \leq |\nabla \eta| = \frac{2}{r\log(k)}. \tag{5}$$

It follows that

$$\begin{aligned}
\int_{B_{\sqrt{k}R}\cap\Sigma}|A|^2 &\overset{(\clubsuit)}{\leq} \int_\Sigma \eta^2|A|^2 \overset{(\diamondsuit)}{\leq} C\int_\Sigma |\nabla_\Sigma \eta|^2 \leq C\int_\Sigma |\nabla\eta|^2 \\
&\overset{(\heartsuit)}{=} C\sum_{l=\frac{\log(k)}{2}+1}^{\log(k)} \int_{(B_{e^lR}\setminus B_{e^{l-1}R})\cap\Sigma} |\nabla\eta|^2 \\
&\overset{(5)}{=} \frac{4C}{(\log(k))^2}\sum_{l=\frac{\log(k)}{2}+1}^{\log(k)} \int_{(B_{e^lR}\setminus B_{e^{l-1}R})\cap\Sigma} \frac{1}{r^2} \\
&\overset{(\sharp)}{\leq} \frac{4C}{(\log(k))^2}\sum_{l=\frac{\log(k)}{2}+1}^{\log(k)} \frac{1}{e^{2l-2}R^2}\mathrm{Vol}((B_{e^lR}\setminus B_{e^{l-1}R})\cap\Sigma) \\
&\leq \frac{4C}{(\log(k))^2}\sum_{l=\frac{\log(k)}{2}+1}^{\log(k)} \frac{1}{e^{2l-2}R^2}\mathrm{Vol}(B_{e^lR}\cap\Sigma) \\
&\overset{(\spadesuit)}{\leq} \frac{4C}{(\log(k))^2}\sum_{l=\frac{\log(k)}{2}+1}^{\log(k)} \frac{2\pi e^{2l}R^2}{e^{2l-2}R^2} \\
&= \frac{4C}{(\log(k))^2}\sum_{l=\frac{\log(k)}{2}+1}^{\log(k)} 2\pi e^2 = \frac{8\pi C e^2}{\log(k)},
\end{aligned}$$

where ($\clubsuit$) uses $\eta\equiv 1$ on $B_{\sqrt{k}R}$, ($\diamondsuit$) uses Lemma 1, ($\heartsuit$) uses $\nabla\eta$ has support between $B_{\sqrt{k}R}$ and B_{kR}, ($\sharp$) uses $r^2\geq e^{2(l-1)}R^2$ on $B_{e^lR}\setminus B_{e^{l-1}R}$ and finally ($\spadesuit$) uses Exercise Sheet 1 (for simplicity we assumed $\frac{\log(k)}{2}$ is an integer). ■

Here is now the main result.

Theorem 1 (Theorem 1.21 in [CM11], Bernstein)**.** *If $u:\mathbb{R}^2\to\mathbb{R}$ is an entire solution to the minimal surface equation, then $u(x,y)=ax+by+c$ for some constants $a,b,c\in\mathbb{R}$.*

Proof. The set $\mathbb{R}^2\times\mathbb{R}=\mathbb{R}^3$ contains B_k for any $k>1$ and so we can apply Corollary 1 with $R=1$ to obtain

$$\int_{B_{\sqrt{k}}\cap\mathrm{Graph}_u}|A|^2\leq\frac{C}{\log(k)},$$

for some constant C. Letting $k\to\infty$, we conclude $|A|^2\equiv 0$, because the left-hand side is nonnegative. But $|A|^2=A_{xx}^2+2A_{xy}^2+A_{yy}^2\overset{(\heartsuit)}{=}\frac{u_{xx}^2+2u_{xy}^2+u_{yy}^2}{1+|\nabla u|^2}$, where ($\heartsuit$) uses the equation (1.92) in [CM11]. Hence $u_{xx}=u_{xy}=u_{yy}=0$, which implies $u(x,y)=ax+by+c$ for some constants $a,b,c\in\mathbb{R}$. This concludes the proof. ■