Monotonicity formula

Philipp Reinke

We follow [Whi16, pages 6 and 7] with an interlude from additional literature.

1 Introduction

So far we were interested in minimal surfaces, i.e. surfaces in $\mathbb{R}^3$ that satisfy $H \equiv 0$. Now, we want to generalise minimality to submanifolds of $\mathbb{R}^n$ with any dimension smaller or equal to n. To do so we need to define the mean curvature of a submanifold in $\mathbb{R}^n$. Let us postpone it to the next section and assume it to be given for a moment.

Definition. A submanifold of $\mathbb{R}^n$ is called a *minimal* (or *stationary*) if its mean curvature is everywhere 0.

Of course, all the minimal surfaces that occurred so far are minimal submanifolds. The main theorem of this talk goes as follows.

Theorem 1 (Monotoncity formula). *Let M be an m-dimensional minimal submanifold of $\mathbb{R}^n$ and let $p \in \mathbb{R}^n$. Then*

$$\Theta(M, p, r) := \frac{\operatorname{area}(M \cap \mathrm{B}(p, r))}{\omega_m r^m}$$

is an increasing function in $r \in (0, \operatorname{dist}(p, \partial M)]$. Indeed,

$$\frac{d}{dr}\ \Theta(M, p, r) \geqslant 0$$

with equality if and only if M intersects $\partial\mathrm{B}(p, r)$ orthogonally.

Here, ω_m denotes the m-dimensional volume of the unit ball in $\mathbb{R}^m$ and $\mathrm{B}(p, r)$ the n-dimensional ball of radius r around p.

What is $\Theta(M, p, r)$ intuitively? It is called the *density ratio* of M in $\mathrm{B}(p, r)$. Let us consider some examples.

Example. The plane $\mathbb{R}^2 \times 0 \subseteq \mathbb{R}^3$ has $\Theta(M, 0, r) = 1$ for all $r > 0$, the unit ball $\mathrm{B} \subseteq \mathbb{R}^3$ has $\Theta(M, 0, r) = 1$ for $r \in (0, 1]$ and two orthogonally intersecting planes in $\mathbb{R}^3$ have $\Theta(M, p, r) = 2$ for p on the intersection.

Note that Θ gives rise to a scale invariant quantity. Consider for $p \in M \backslash \partial M$ the so-called *density of M at p*

$$\Theta(M, p) := \lim_{r \to 0} \Theta(M, p, r).$$

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2 Interlude: The geometry of submanifolds

What follows is taken from [CM11, Section 1.2], [Car91, Chapters 1 and 6] and [Lee11, Chapters 2 and 5]. This interlude/digression is more general than needed for our main theorem as we only consider submanifolds of $\mathbb{R}^n$. However, this is how the mean curvature is introduced in [CM11, Section 1.2] and it also provides a greater insight. We do not give proofs and focus on a general understanding. Throughout this section M denotes a smooth manifold.

Riemannian manifolds

We want to do geometry on a manifold. So, we need notions like distances and angles. This leads to the following definition.

Definition. A *Riemannian metric* g on M is a smoothly varying family of inner products on the tangent spaces of M. Namely, g associates to each $p \in M$ a positive definite, symmetric bilinear form on T_pM

$$g_p : \mathrm{T}_pM \times \mathrm{T}_pM \to \mathbb{R}$$

and smoothness means that the function

$$p \in M \mapsto g_p(X_p, Y_p) \in \mathbb{R}$$

is smooth for every locally defined smooth vector fields X, Y in M. A pair (M, g) is called a *Riemannian manifold*.

Example (Euclidean $\mathbb{R}^n$). The canonical Riemannian metric on $\mathbb{R}^n$ is given by the usual dot-product in $\mathbb{R}^n$, i.e. $g_p(v, w) = v \cdot w$ at every $p \in \mathbb{R}^n$ and for $v, w \in \mathrm{T}_p\mathbb{R}^n \cong \mathbb{R}^n$.

The Levi-Civita connection

We denote by $\mathfrak{X}(M)$ the set of all smooth vector fields on M. Given a vector field $X \in \mathfrak{X}(M)$ and a real-valued function $f \in C^\infty(M)$ we define a new vector field $fX : M \to \mathrm{T}M$ via

$$(fX)_p := f(p)X_p.$$

Definition. A *connection* in TM is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ written $(X, Y) \mapsto \nabla_X Y$ satisfying

(i) $\nabla_X Y$ is linear over $C^\infty(M)$ in X, i.e $\quad \nabla_{f_1X_1+f_2X_2}Y = f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y$

(ii) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y, i.e $\quad \nabla_X(a_1Y_1 + a_2Y_2) = a_1\nabla_X Y_1 + a_2\nabla_X Y_2$

(iii) ∇ satisfies the Leibniz/product rule, i.e. $\quad \nabla_X(fY) = f\ \nabla_X Y + (Xf)Y$

where $X_1, X_2, X, Y \in \mathfrak{X}(M)$, $f_1, f_2, f \in C^\infty(M)$ and $a_1, a_2 \in \mathbb{R}$. We call $\nabla_X Y$ the *covariant derivative of* Y *in the direction* X.

There are several connections on a Riemannian manifold and we need to single out one.

Definition/Lemma ([Lee11, Theorem 5.10]). Given a Riemannian manifold (M, g) there is a unique connection, called the *Levi-Civita connection of* g, such that ∇

(i) is *symmetric*, i.e. $\quad \nabla_X Y - \nabla_Y X = [X, Y]$

(ii) is *compatible with* g, i.e. $\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

What is the intuition behind these definitions and why are they how they are? One can think of $\nabla_X Y$ as a generalised directional derivative. The advantage of this axiomatic approach is that it does not depend on coordinates and these axioms are designed to capture the main properties of the Levi-Civita connection in the Euclidean $\mathbb{R}^n$ which goes as follows.

Example (Levi-Civita connection in the Euclidean $\mathbb{R}^n$). Let X and $Y = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ be two vector fields on $\mathbb{R}^n$ where $f_i : \mathbb{R}^n \to \mathbb{R}$ are the coordinate functions of Y. The Levi-Civita connection is then

$$\nabla_X Y = \sum_{i=1}^n \nabla_X f_i \, \frac{\partial}{\partial x_i},$$

where $\nabla_X f_i$ denotes the usual directional derivative, i.e. $(\nabla_X f_i)(x) := \nabla_{X(x)} f_i(x)$. Note that some use instead of ∇ the symbol ∂ or D for the usual directional derivative.

Submanifold geometry

Let $F : M \to \widetilde{M}$ be an immersion of a smooth m-dimensional manifold M into a a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ with Levi-Civita connection $\widetilde{\nabla}$. Then $\widetilde{g}$ induces a Riemannian metric g on M via

$$g_p(v,w) := \widetilde{g}_{F(p)}(dF(p)(v), dF(p)(w)) \quad \text{for all } p \in M \text{ and } v, w \in \mathrm{T}_pM.$$

We call g the *metric induced by* F.

Example. If $S \subseteq \mathbb{R}^3$ is a surface in the Euclidean $\mathbb{R}^3$ the induced metric with respect to the inclusion is exactly the first fundamental form.

The induced metric splits $\mathrm{T}_p\widetilde{M}$ as an orthogonal direct sum

$$\mathrm{T}_p\widetilde{M} = \mathrm{T}_pM \ \oplus \ (\mathrm{T}_pM)^\perp \quad \text{where } (\mathrm{T}_pM)^\perp := \{v \in T_p\widetilde{M} \mid g_p(v,w) = 0 \text{ for all } w \in \mathrm{T}_pM\}$$

for any $p \in \widetilde{M}$ and we write for $v \in \mathrm{T}_p\widetilde{M}$

$$v = v^T + v^N \quad \text{with } v^T \in T_pM \text{ and } v^N \in (T_pM)^\perp.$$

Moreover, one can see that the Levi-Civita connection on M, denoted ∇ or ∇_M, is

$$\nabla_X Y = (\nabla_M)_X Y = (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y})^T \quad \text{where } X, Y \in \mathfrak{X}(M) \text{ and } \widetilde{X}, \widetilde{Y} \text{ are local extensions to } \widetilde{M}.$$

One can check that this is independent from the choice of the local extension.

Let $U \subseteq M$ be an open set and denote by $\mathfrak{X}(U)^\perp$ the set of vector fields that are normal to $F(M) \subseteq \widetilde{M}$. Consider the map $A : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)^\perp$

$$A(X,Y) := (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y})^N = \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} - \nabla_X Y \quad \text{for } X, Y \in \mathfrak{X}(U).$$

One can check that A is symmetric and bilinear (cf. [Car91, Proposition 2.1 on page 127]). We define the *mean curvature* to be

$$H := \sum_{i=1}^m A(E_i, E_i) = \sum_{i=1}^m (\widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i)^N \quad \text{where } (E_1, ..., E_m) \text{ is an orthonormal frame of } \mathrm{T}M.$$

Here, an *orthonormal frame* is an ordered m-tuple of smooth vector fields (E_i) defined on an open set $U \subseteq M$ such that at every $p \in U$, $(E_i(p))$ is an orthonormal basis of T_pM. The definition of H is independent from the choice of (E_i) (cf. [Car91, p. 133]).

3 Proof of the Monotonicity formula

Back to the Euclidean $\mathbb{R}^n$. To begin with we need three technical theorems. The first has already been used but we state it again taken from [CM11, 3.2 on page 24].

THEOREM 2 (Coarea formula). *Let Σ be a submanifold of $\mathbb{R}^n$ and $h : \Sigma \to \mathbb{R}$ a proper (i.e. $h^{-1}((-\infty, t])$ is compact for all $t \in \mathbb{R}$) Lipschitz function on Σ. Then for every locally integrable function f on Σ and $t \in \mathbb{R}$*

$$\int_{\{h \leqslant t\}} f|\nabla_\Sigma h| = \int_{-\infty}^{t} \int_{h=\tau} f \; d\tau$$

where $\nabla_\Sigma h$ is the tangential projection of the gradient.

THEOREM 3 (Generalised Divergence Theorem). *Let M be a compact m-dimensional manifold in $\mathbb{R}^n$ and X a smooth vector field on $\mathbb{R}^n$*

$$\int_M \operatorname{div}_M X \; dS = \int_{\partial M} X \cdot \nu_{\partial M} \; ds - \int_M H \cdot X \; dS$$

where $\nu_{\partial M}(x)$ is the unit vector in the tangent plane $\mathrm{T}_x M$ normal to ∂M such that it points away from M and

$$\operatorname{div}_M X := \sum_{i=1}^{m} E_i \cdot \nabla_{E_i} X$$

for any orthonormal frame $(E_1, \ldots, E_m)$ of $\mathrm{T}M$.

Proof of Theorem 3. We split $X = X^T + X^N$ into its normal and tangential component and apply the ordinary divergence theorem

$$\begin{aligned} \int_M \operatorname{div}_M X \; dS &= \int_M \operatorname{div}_M X^T \; dS \; + \int_M \operatorname{div}_M X^N \; dS \\ &= \int_{\partial M} X^T \cdot \nu_{\partial M} \; ds + \int_M \operatorname{div}_M X^N \; dS \\ &= \int_{\partial M} X \cdot \nu_{\partial M} \; ds \; + \int_M \operatorname{div}_M X^N \; dS \end{aligned}$$

where the last holds because of $X^T \cdot \nu_{\partial M} = -X^N \cdot \nu_{\partial M} + X \cdot \nu_{\partial M} = X \cdot \nu_{\partial M}$. So, we are left to show $\operatorname{div}_M X^N = H \cdot X$.

$$\begin{aligned} \operatorname{div}_M X^N &\overset{\text{def}}{=} \sum_{i=1}^{m} E_i \cdot \nabla_{E_i}(X^N) = \sum_{i=1}^{m} \nabla_{E_i}(E_i \cdot X^N) - (\nabla_{E_i} E_i) \cdot X^N = \sum_{i=1}^{m} \; 0 \; - (\nabla_{E_i} E_i)^N \cdot X \\ &\overset{\text{def}}{=} -H \cdot X. \end{aligned}$$

The second equality holds by the compatibility with the metric. The third as X^N is orthogonal to $\mathrm{T}_x \Sigma$ (therefore $E_i \cdot X^N = 0$) and since for every $v, w \in \mathbb{R}^n$

$$v \cdot w^N = v^T \cdot w^N + v^N \cdot w^N = 0 + v^N \cdot w^N = v^N \cdot w - v^N \cdot w^T = v^N \cdot w - 0 \cdot w = v^N \cdot w.$$

□

THEOREM 4. *Let M be a compact m-dimensional minimal submanifold of $\mathbb{R}^n$. Then*

$$m\ \mathrm{area}(M) = \int_{x\in\partial M} x\cdot\nu_{\partial M}\ ds.$$

Proof of Theorem 4. We apply Theorem 3 to the vector field $X(x) = x$. In this case

$$\mathrm{div}_M(X) \equiv m \quad \text{since } \nabla_v X = v \text{ for all } v\in\mathbb{R}^n \text{ and hence } \sum_i (E_i\cdot\nabla_{E_i}X) = \sum_i |E_i|^2 = m$$

and $H = 0$ by minimality of M. (Note: $\nabla_v X$ is just the Levi-Civita connection in $\mathbb{R}^n$ with the constant vector field v.) □

Now, we are in good shape to proof our main theorem.

Proof of Theorem 1. We may assume $p = 0$ and define $M_r := M\cap \mathrm{B}(0,r)$ with $\partial M_r = M\cap\partial\mathrm{B}(0,r)$ due to $r\in(0,\mathrm{dist}(0,\partial M)]$. Consider

$$A(r) := m\text{-dimensional area of } M_r \quad \text{and} \quad L(r) := (m-1)\text{-dimensional measure of } \partial M_r.$$

We will show the following inequalities

$$\text{(i) } A'(r) \geqslant L(r) \qquad \text{(ii) } rL(r)\geqslant mA(r)$$

with equality in (i) and (ii) if and only if M intersects $\partial\mathrm{B}(0,r)$ orthogonally. From this we conclude

$$A' - \frac{m}{r}A \geqslant 0 \quad\Rightarrow\quad \frac{1}{r^m}A' - \frac{m}{r^{m+1}}A\geqslant 0 \quad\Rightarrow\quad \left(\frac{A}{r^m}\right)'\geqslant 0$$

with equality if and only if M intersects $\partial\mathrm{B}(0,r)$ orthogonally.

<u>ad(i)</u> We apply Theorem 2 to $\Sigma = M$ with $h(x) := |x|$ and $f\equiv 1$ and $t = r\in(0,\mathrm{dist}(0,\partial M)]$

$$\frac{d}{dr}\int_{\{h\leqslant r\}} |\nabla_\Sigma h| = \int_{\{h=r\}} d\tau.$$

We see that $\{h\leqslant r\} = M_r$ and $\{h = r\} = \partial M_r$. Hence

$$\frac{d}{dr}\int_{M_r} |\nabla_\Sigma h| = L(r) \quad\Rightarrow\quad \frac{d}{dr}A(r)\geqslant L(r)$$

where we use $|\nabla_\Sigma h|\leqslant 1$ with equality if and only if M intersects $\partial\mathrm{B}(0,r)$ orthogonally. Indeed, orthogonal intersection $\Leftrightarrow$ $\nabla h(x)\in \mathrm{T}_x M$ for all $x\in\partial\mathrm{B}(0,r)$ $\Leftrightarrow$ $|\nabla_\Sigma h(x)| = 1$ for all $x\in\partial\mathrm{B}(0,r)$.

<u>ad(ii)</u> By Theorem 4

$$mA(r) = \int_{x\in\partial M_r} x\cdot\nu_{\partial M}\ ds \leqslant \int_{x\in\partial M_r} |x|\ ds = \int_{x\in\partial M_r} r\ ds = rL(r)$$

where the inequality is due to the Cauchy-Schwarz inequality with equality if and only if all $x\in\partial M_r$ are orthogonal to $\nu_{\partial M}$ which is the case if and only if M intersects $\partial\mathrm{B}(0,r)$ orthogonally.

□

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