Talk “Stability Inequality”

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The talk follows [CM11], pages 28-30.

1 Motivation

The goal of this talk and the one after it is to prove that if a graph over $\mathbb{R}^2$ is a minimal surface then it is a plane. More formally we want to prove the following theorem due to Bernstein.

Theorem. *If $u\colon \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation, then $u(x,y) = ax + by + c$ for some constants $a, b, c \in \mathbb{R}$.*

2 Preliminaries

In this section we calculate the metric, curvature and second fundamental form of a graph.

Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a surface. Recall the Gauss map

$$N\colon \Sigma \to S^2 \subseteq \mathbb{R}^3$$

is a continuous unit normal. For a (connected) surface there are precisely two such choices. If $\{E_1(q), E_2(q)\}$ are an orthonormal basis of the tangent space $T_q\Sigma$ for $q \in \Sigma$, then N is given by

$$N(q) = \frac{E_1(q) \wedge E_2(q)}{|E_1(q) \wedge E_2(q)|}.$$

In a chart (U, φ) we called $\nu = N \circ \varphi$ the Gauss map. Note that $\{E_1(q), E_2(q)\}$ is also an orthonormal basis of the tangent space $T_{N(q)}S^2$ as $N(q)$ is normal to S^2 at $N(q)$ and thus

$$dN_q\colon T_q\Sigma \to T_{N(q)}S^2$$

is uniquely defined by

$$A_{ij} := \langle dN(E_i), E_j \rangle,\ i = 1, 2.$$

Recall that dN is symmetric, hence diagonalisable over an orthonormal basis $\{E_1, E_2\}$, called *principal directions*, and with eigenvalues λ_1, λ_2, called *principal curvatures*, and that $H \coloneqq \frac{1}{2}\operatorname{tr}(dN) = \frac{1}{2}(\lambda_1 + \lambda_2)$ and $K \coloneqq \det(dN) = \lambda_1\lambda_2$.

Let now $U \subseteq \mathbb{R}^2$ be an open domain, $f\colon U \to \mathbb{R}$ be a C^2-function. Then

$$\Sigma = \operatorname{graph}(f) = \{(x, y, f(x, y)) \mid (x, y) \in U\} \subseteq \mathbb{R}^3$$

is a surface with global chart

$$\varphi\colon U \to \mathbb{R}^3;\ (x, y) \mapsto (x, y, f(x, y)).$$

Recall from Exercise Sheet 2 point e), that if a graph satisfies the minimal surface equation, then $H = 0$ (actually this is an if and only if statement). So $\lambda_1 = -\lambda_2$ and hence the Hilbert–Schmidt norm of dN satisfies

$$|dN|^2 = |A|^2 = \lambda_1^2 + \lambda_2^2 = -2\lambda_1\lambda_2 = -2K = -2\det(dN). \tag{1}$$

Notice that this implies that the Gauss curvature K satisfies $K \leq 0$.

Furthermore, recall from Exercise Sheet 1 that

$$N(x, y, z) = \frac{(-f_x(x, y), -f_y(x, y), 1)}{\sqrt{1 + |\nabla f(x, y)|^2}},\ (x, y, z) \in U \times \mathbb{R}$$

is the upward pointing unit normal. Indeed, $|N| = 1$ and a basis of the tangent plane is given by the vectors

$$\varphi_x = (1, 0, f_x),\ \varphi_y = (0, 1, f_y)$$

which are both orthogonal to N.

Recall from Exercise Sheet 2 the calculation of the first fundamental form I given by the matrix g:

$$g_{xx} = (1 + f_x^2),\ g_{xy} = g_{yx} = f_x f_y,\ g_{yy} = (1 + f_y^2)$$

with inverse matrix g^{-1}:

$$g^{xx} = \frac{1 + f_y^2}{1 + |\nabla f|^2},\ g^{xy} = g^{yx} = \frac{-f_x f_y}{1 + |\nabla f|^2},\ g^{yy} = \frac{1 + f_x^2}{1 + |\nabla f|^2}$$

and of the Gauss curvature K:

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + |\nabla f|^2)^2}.$$

Recall furthermore that the area element of the graph Σ is given by

$$d\,\text{Area} = d\sigma = \sqrt{1 + |\nabla f|^2} dx \wedge dy$$

so that

$$K d\sigma = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + |\nabla f|^2)^{3/2}} dx \wedge dy.$$

2.1 Differential Forms

In this section we present some facts on differential forms used in the next section, which can all be found in Prof. Will Merry's lecture notes on Differential Geometry (see https://www.merry.io/courses/differential-geometry/) or in any introductory textbook in differential geometry. The idea behind differential forms is to generalise integration from $\mathbb{R}^n$ (and in some cases submanifolds) to general manifolds.

A differential k-form on a manifold M is an alternating C^∞ k-multilinear function

$$\mathfrak{X}(M)^k \to C^\infty(M).$$

So at every point, a differential k-form ω takes in k tangent vectors and produces a real number, and it does so in an alternating and smooth way. The number k is called the degree of ω, and the space of differential k-forms is denoted by $\Omega^k(M)$, and $\Omega(M) \coloneqq \bigoplus_{k=0}^{\infty} \Omega^k(M)$.

The vector space $\Omega(M)$ is in addition turned into a graded algebra by the following wedge product:

$$\wedge \colon \Omega(M) \times \Omega(M) \to \Omega(M),\ (\omega, \theta) \mapsto \omega \wedge \theta$$

which satisfies $\omega \wedge \theta \in \Omega^{k+l}(M)$ for $\omega \in \Omega^k(M)$ and $\theta \in \Omega^l(M)$.

From $\omega \in \Omega^k(M)$ one can produce a new differential form, now of degree $k+1$ using the exterior differential $d \colon \Omega(M) \to \Omega(M)$. The exterior differential satisfies:

- If $\omega \in \Omega^k(M)$ and $\theta \in \Omega^l(M)$ then $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$
- If f is a differential 0-form, i.e. a smooth function, then df is the normal differential of f.
- For every differential form ω it holds $d(d\omega) = 0$.

Given a smooth map $\phi \colon M \to N$ between manifolds and a differential k-form $\omega \in \Omega^k(N)$ one can define the pullback $\phi^*\omega \in \Omega^k(M)$ by

$$\phi^*\omega(X_1, \ldots, X_k) = \omega(d\phi(X_1), \ldots, d\phi(X_k)).$$

It can be shown that the exterior differential commutes with pullbacks, i.e. that $d(\phi^*\omega) = \phi^*(d\omega)$.

A differential form ω is called *closed* if $d\omega = 0$. It is called *exact* if there exists a differential form θ so that $\omega = d\theta$. It is clear that every exact form is closed. However, the Poincaré Lemma states that the converse is true on contractible manifolds M.

Finally, Stokes' theorem will also be needed.

Theorem (Stokes)**.** *Let M be an n-dimensional oriented manifold with boundary ∂M and let ω be a differential $(n-1)$-form with compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

.

In particular, if ω vanishes on ∂M then the integrals are both zero.

3 The Stability Estimate

In this section we prove an estimate on the integral of the curvature which will be used in the proof of Bernstein's theorem.

Lemma. *If $f\colon U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a solution of the minimal surface equation, then for all nonnegative Lipschitz functions $\eta\colon \mathbb{R}^3 \to \mathbb{R}$ with support contained in $U \times \mathbb{R}$,*

$$\int_{\mathrm{graph}(f)} |A|^2 \eta^2 d\sigma \leq C \int_{\mathrm{graph}(f)} |\nabla_{\mathrm{graph}(f)} \eta|^2 d\sigma$$

for a constant $C \in \mathbb{R}$ independent of U, f and η.

Proof. Let $\omega = d\sigma_{S^2}$ be the area form on the sphere S^2. Note that the Gauss map N takes values in the upper hemisphere, which is a contractible space. By the Poincaré Lemma, there exists a one-form α on the upper hemisphere so that $\omega = d\alpha$ (on the upper hemisphere). By (1), the change of variables formula and because the differential and pullback commute we get the following chain of equations:

$$|A|^2 d\sigma = -2K d\sigma = -2\det(dN) d\sigma = 2N^*\omega = 2d(N^*\alpha) \tag{2}$$

As α is a one-form there exists a constant $C_\alpha \in \mathbb{R}$ so that

$$|N^*\alpha| \leq C_\alpha |dN| = C_\alpha |A|. \tag{3}$$

In fact,

$$|N^*\alpha| = \sup_{|X|=1} |N^*\alpha(X)| = \sup_{|X|=1} |\alpha(dN(X))| \leq |\alpha||A|.$$

Using (2) we get

$$\int_\Sigma \eta^2 |A|^2 d\sigma = 2\int_\Sigma \eta^2 d(N^*\alpha) = 2\int_\Sigma d(\eta^2 N^*\alpha) - 4\int_\Sigma \eta d\eta \wedge N^*\alpha.$$

By Stokes' theorem and by assumption on η, the first integral vanishes. Hence, using (3) we get

$$\begin{aligned}\int_\Sigma \eta^2 |A|^2 d\sigma &= -4\int_\Sigma \eta d\eta \wedge N^* \leq 4C_\alpha \int_\Sigma \eta |\nabla_\Sigma \eta||A| d\sigma \\ &\leq 4C_\alpha \left(\int_\Sigma \eta^2 |A|^2 d\sigma\right)^{1/2} \left(\int_\Sigma |\nabla_\Sigma \eta|^2 d\sigma\right)^{1/2} \end{aligned} \tag{4}$$

where the last inequality follows by the Cauchy–Schwarz inequality. Notice that we used also that

$$-4\int_\Sigma \eta d\eta \wedge N^*\alpha \le 4\int_\Sigma |\eta||\nabla_\Sigma \eta||N^*\alpha|d\sigma$$

by

$$\sup_{X\in T\Sigma,\ |X|=1} |d\eta(X)| = |\nabla_\Sigma \eta|.$$

Squaring (4) and dividing both sides by $\int_\Sigma \eta^2|A|^2 d\sigma$ gives

$$\int_\Sigma \eta^2|A|^2 d\sigma \le 16C_\alpha^2 \int_\Sigma |\nabla_\Sigma \eta|^2 d\sigma,$$

as desired. □