Mathematical Foundations for Finance

Exercise sheet 12

Please hand in your solutions until Tuesday, 10/12/2019, 18:00 into your assistant's box next to HG G 53.2.

Exercise 12.1 In this exercise, we show various results that are frequently used in stochastic analysis. Some of them were given as hints in the previous exercises.

(a) Let X be an RCLL $\mathbb{F}$-adapted stochastic process and τ an $\mathbb{F}$-stopping time. Show that if X^τ is an $\mathbb{F}$-martingale, then so is X^σ for any $\mathbb{F}$-stopping time σ with $\sigma \leq \tau$ P-a.s.
Hint: You can use the result that a stopped RCLL martingale is again an RCLL martingale. This is similar to the result you have proved in Exercise 3.1 (c).

(b) Let M and N be two RCLL local $\mathbb{F}$-martingales. Show that the linear combination $\alpha M + \beta N$ for any $\alpha, \beta \in \mathbb{R}$ is an RCLL local $\mathbb{F}$-martingale as well.
Hint: Make use of the result in (a).

(c) We say that two Brownian motions W^1 and W^2 on the same probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ are *correlated with correlation* $\rho \in [-1, 1]$ if for $s \leq t$, the increments $W_t^1 - W_s^1$ and $W_t^2 - W_s^2$ are independent of $\mathcal{F}_s$ and jointly normally distributed with $\mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}.$$

Show that $[W^1, W^2]_t = \rho t$ P-a.s.
Hint: Define $B^\lambda = \lambda(W^1 + W^2)$ with $\lambda \in \mathbb{R}$. Find λ such that B^λ becomes a $(P, \mathbb{F})$-Brownian motion. Then compute $[B^\lambda]$ in terms of W^1 and W^2, using the properties of $[\,\cdot\,,\,\cdot\,]$.

Exercise 12.2 Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale null at 0. We define the process

$$L := \mathcal{E}(X) := e^{X - \frac{1}{2}[X]}.$$

(a) Show via Itô's formula that

$$L_t = 1 + \int_0^t L_s dX_s, \quad \forall t \geq 0. \tag{1}$$

Conclude that L is a continuous local martingale if and only if X is a continuous local martingale.

(b) Show that $L = \mathcal{E}(X)$ is the only solution to (1) for a given X.
Hint: Let L' be another solution of (1). Compute $\frac{L'}{L}$ using Itô's formula.

(c) Let $Y = (Y_t)_{t\geq 0}$ be another continuous semimartingale null at 0. Show *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}\left(X + Y + [X, Y]\right).$$

Exercise 12.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtered probability space and consider two *independent* Brownian motions $W^1 = (W_t^1)_{t\in[0,T]}$ and $W^2 = (W_t^2)_{t\in[0,T]}$. Let $\widetilde{S}^1 = (\widetilde{S}_t^1)_{t\in[0,T]}$ and $\widetilde{S}^2 = (\widetilde{S}_t^2)_{t\in[0,T]}$ be two *undiscounted* stock price processes with the dynamics

$$d\widetilde{S}_t^1 = \widetilde{S}_t^1 \left(\mu_1 dt + \sigma_1 dB_t^1\right), \quad \widetilde{S}_0^1 > 0,$$
$$d\widetilde{S}_t^2 = \widetilde{S}_t^2 \left(\mu_2 dt + \sigma_2 dB_t^2\right), \quad \widetilde{S}_0^2 > 0,$$

where $B^1 = W^1$, $B^2 = \alpha W^1 + \sqrt{1-\alpha^2} W^2$, for some $\alpha \in [0,1)$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.

(a) Find the SDEs satisfied by $X^1 := \frac{\widetilde{S}^2}{\widetilde{S}^1}$ and $X^2 := \frac{\widetilde{S}^1}{\widetilde{S}^2}$.
Remark: Since $\widetilde{S}^1$ and $\widetilde{S}^2$ have continuous trajectories and satisfy $\widetilde{S}^1_t, \widetilde{S}^2_t > 0$ for all $t \in [0,T]$ P-a.s., we can choose each of them as *numéraire*.

(b) For $\beta_1, \beta_2 \in \mathbb{R}$, define the continuous local $(P, \mathbb{F})$-martingale $L^{(\beta_1,\beta_2)} := \beta_1 W^1 + \beta_2 W^2$. Show that for all $\beta_1, \beta_2 \in \mathbb{R}$, the stochastic exponential $Z^{(\beta_1,\beta_2)} := \mathcal{E}(L^{(\beta_1,\beta_2)})$ is a true $(P, \mathbb{F})$-martingale on $[0,T]$.

(c) For $\beta_1, \beta_2 \in \mathbb{R}$, define by $dQ^{(\beta_1,\beta_2)} = Z_T^{(\beta_1,\beta_2)} dP$ a probability measure $Q^{(\beta_1,\beta_2)}$ which is equivalent to P on $\mathcal{F}_T$. Fix $\beta_1, \beta_2 \in \mathbb{R}$. Using Girsanov's theorem, show that the two processes $\widetilde{W}^1_t := W^1_t - \beta_1 t$ and $\widetilde{W}^2_t := W^2_t - \beta_2 t$, $t \in [0,T]$, are local $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales. Conclude that

$$\widetilde{B}^1 := \widetilde{W}^1 \quad \text{and} \quad \widetilde{B}^2_t := B^2_t - (\alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2)t, \quad t \in [0,T],$$

are local $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales as well.
Remark: One can show that $\widetilde{W}^1$ and $\widetilde{W}^2$ are *independent* Brownian motions under $Q^{(\beta_1,\beta_2)}$ and correspondingly that $\widetilde{B}^1$ and $\widetilde{B}^2$ are *correlated* Brownian motions under $Q^{(\beta_1,\beta_2)}$.

(d) What conditions on $\beta_1, \beta_2 \in \mathbb{R}$ make the processes X^1 and X^2 $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales?