

Mathematical Foundations for Finance

Exercise sheet 6

Please hand in your solutions until Tuesday, 29/10/2019, 18:00 into your assistant's box next to HG G 53.2.

Exercise 6.1 Consider a financial market in finite discrete time on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with undiscounted prices \tilde{S}^0, \tilde{S} and discounted prices $1, S = \tilde{S}/\tilde{S}^0$. An arbitrage opportunity in the undiscounted market is a self-financing strategy φ with $\tilde{V}(\varphi) \geq -a$ P -a.s. for some $a \geq 0$ (admissibility), $\tilde{V}_0(\varphi) = 0$, $\tilde{V}_T(\varphi) \geq 0$ P -a.s. and $P[\tilde{V}_T(\varphi) > 0] > 0$.

- (a) Show that (\tilde{S}^0, \tilde{S}) is free of arbitrage if and only if (S^0, S) is.
Hint: Remember that in finite discrete time we have that $NA \iff NA'$.
- (b) Construct an example where \tilde{S} admits an EMM, but is not arbitrage-free. Does S then admit an EMM? What can you say about \tilde{S}^0 for any such example?
- (c) In your example, construct explicitly an arbitrage opportunity for the undiscounted market.
- (d) Try to provide some intuition behind the existence of an EMM for \tilde{S} not implying (NA) when we know that the existence of an EMM for S does.

Exercise 6.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P, \tilde{S}^0, \tilde{S}^1)$ be our canonical setup for a one-period trinomial model in which the evolution of $(\tilde{S}^0, \tilde{S}^1)$ is given by

$$\tilde{S}_0^1 = S_0^1 = 80, \quad \tilde{S}_1^1 = \begin{cases} 120 & \text{with probability } p_1 = 0.2, \\ 90 & \text{with probability } p_2 = 0.3, \\ 60 & \text{with probability } p_3 = 0.5 \end{cases}$$
$$\tilde{S}_0^0 = 1, \quad \tilde{S}_1^0 = 1 + 0.05.$$

- (a) Check if the market is arbitrage-free by finding at least one EMM for $S^1 = \tilde{S}^1/\tilde{S}^0$.
- (b) Find the set of all EMMs for S^1 .
- (c) Compute $E_Q \left[\frac{\tilde{C}}{1+0.05} \right]$ for all $Q \in P_e(S^1)$, where \tilde{C} is the (undiscounted) payoff of a European call option with maturity $T = 1$ and strike price $\tilde{K} = 80$, i.e. $\tilde{C}(\omega) = (\tilde{S}_1^1(\omega) - 80)^+$.
- (d) Determine whether \tilde{C} as given in (c) is attainable.
- (e) Find the set of all attainable payoffs $\tilde{H} \in L_+^0(\mathcal{F}_T)$.
Hint: Every payoff is characterized by the values it takes on the atoms of \mathcal{F}_T . The set of all attainable payoffs can be identified with the set of solutions to a linear system.

Exercise 6.3 Consider the discounted market $(\Omega, \mathcal{F}, \mathbb{F}, P, 1, S^1)$ and assume that the stock price process is adapted to \mathbb{F} . Following points (a)–(c), show that $P_e(S^1)$, the set of all EMMs for S^1 , is convex, i.e. that for all $Q_1, Q_2 \in P_e(S^1)$, the map $Q^\lambda : \mathcal{F} \rightarrow \mathbb{R}$ given by

$$Q^\lambda[A] = \lambda Q_1[A] + (1 - \lambda) Q_2[A] \quad \text{for } A \in \mathcal{F}$$

is an EMM for S^1 for all $\lambda \in [0, 1]$.

- (a) Show that Q^λ is a probability measure and that it is equivalent to P for all $\lambda \in [0, 1]$.
- (b) Fix a $\lambda \in [0, 1]$. By the Radon–Nikodým theorem (see page 40 in the lecture notes), since Q^λ is a probability measure equivalent to P , there exists a density $\mathcal{D}^\lambda := \frac{dQ^\lambda}{dP} > 0$ P -a.s. such that

$$Q^\lambda[A] = E[\mathcal{D}^\lambda \mathbf{1}_A] \quad \forall A \in \mathcal{F}.$$

Write \mathcal{D}^λ as function of $\mathcal{D}^i := \frac{dQ_i}{dP}$ for $i = 1, 2$, the densities of Q_1 and Q_2 w.r.t. P , respectively, and deduce the form of the density process of Q^λ with respect to P .

- (c) Conclude that Q^λ is an equivalent martingale measure for S^1 for each $\lambda \in [0, 1]$.