Coordinator
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Mathematical Foundations for Finance

Exercise sheet 8

Please hand in your solutions until Tuesday, 12/11/2019, 18:00 into your assistant's box next to HG G 53.2.

Exercise 8.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion (BM) defined on some probability space $(\Omega, \mathcal{F}, P)$ (without filtration). Show that

(a) $W^1 := -W$ is a BM.

(b) $W_t^2 := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha B + \sqrt{1-\alpha^2} B'$ is a BM, where B and B' are two independent BMs and $\alpha \in [0, 1]$.

(d) Show that the independence of B and B' in (c) cannot be omitted, i.e., if B and B' are *not* independent, then W^3 need not be a BM. Give two examples.

Exercise 8.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion. Let $a, b > 0$ and define

$$\tau_a = \inf\{t \geq 0 \,|\, W_t > a\},$$
$$\sigma_{a,b} = \inf\{t \geq 0 \,|\, W_t > a + bt\}.$$

(a) Show that for $\tau \in \{\tau_a, \sigma_{a,b}\}$ and all $\alpha \in \mathbb{R}$ we have that

$$E\left[e^{\alpha W_{\tau\wedge t} - \frac{1}{2}\alpha^2(\tau\wedge t)}\right] = 1.$$

(b) Using your result from (a) show that

$$e^{\alpha a} E\left[e^{-\frac{1}{2}\alpha^2 \tau_a}\right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform ϕ_{τ_a} of τ_a is given by

$$\phi_{\tau_a}(\lambda) := E\left[e^{-\lambda\tau_a}\right] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.
Hint 2: Use that $W_{\tau_a} = a$ P-a.s.; we will show this in another exercise sheet.

(c) Using your result from (a) show that

$$e^{\alpha a} E\left[e^{(ab - \frac{1}{2}\alpha^2)\sigma_{a,b}}\right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform $\phi_{\sigma_{a,b}}$ of $\sigma_{a,b}$ is given by

$$\phi_{\sigma_{a,b}}(\lambda) := E\left[e^{-\lambda\sigma_{a,b}}\right] = e^{-a(b+\sqrt{b^2+2\lambda})}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.
Hint 2: Use that $W_{\sigma_{a,b}} = a + b\sigma_{a,b}$ P-a.s.

(d) Show that τ_a is P-a.s. finite for any $a > 0$ and that $\sigma_{a,b}$ takes the value of $+\infty$ with a positive probability for any $a, b > 0$.

Exercise 8.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some sufficiently rich filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous convex function. Show that if the stochastic process $\big(f(W_t)\big)_{t\geq 0}$ is integrable, then it is a $(P, \mathbb{F})$-submartingale.
Hint: We have done something similar in discrete time.

(b) Given a $(P, \mathbb{F})$-martingale $(M_t)_{t\geq 0}$ and a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, show that the process

$$\big(M_t + g(t)\big)_{t\geq 0}$$

is a $(P, \mathbb{F})$-supermartingale if and only if g is decreasing, and a $(P, \mathbb{F})$-submartingale if and only if g is increasing.

(c) Show that the following stochastic processes are $(P, \mathbb{F})$-submartingales but not martingales:

(i) W^2,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: Use the result from (a) and (b), respectively.

(d) Show that any $(P, \mathbb{F})$-local martingale which is null at 0 and uniformly bounded from below is a $(P, \mathbb{F})$-supermartingale.
Hint: We have done this in discrete time already.