# Mathematical Foundations for Finance

## Exercise sheet 9

*Please hand in your solutions until Tuesday, 19/11/2019, 18:00 into your assistant's box next to HG G 53.2.*

**Exercise 9.1** Let $(Y_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(Y_1, \dots, Y_k)$ for all $k \in \mathbb{N}$. Let $E[Y_k] = \mu$ and $\mathrm{Var}(Y_k) = \sigma^2$ for all $k \in \mathbb{N}$ with $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Define additionally $X = (X_k)_{k\in\mathbb{N}_0}$ by

$$X_k = \sum_{j=1}^{k} Y_j \quad \text{for all } k \in \mathbb{N}_0,$$

and assume that $X$ is adapted to $\mathbb{F}$.

(a) Show that for any $\mathbb{F}$-adapted integrable process $Z = (Z_k)_{k\in\mathbb{N}_0}$, there exists a $P$-a.s. unique decomposition of $Z$ into $Z = M + A$ with $M = (M_k)_{k\in\mathbb{N}_0}$ a $(P, \mathbb{F})$-martingale and $A = (A_k)_{k\in\mathbb{N}_0}$ an $\mathbb{F}$-predictable integrable process with $A_0 = 0$.
*Hint: This is the Doob decomposition. Show the existence by construction.*

(b) Using (a), explicitly derive the processes $M$ and $A$ in the Doob decomposition of $X$.

(c) Explicitly derive the optional quadratic variation $[M] = \big([M]_k\big)_{k\in\mathbb{N}_0}$ of the square-integrable martingale $M$ from (b), and show that $M^2 - [M]$ is a martingale.
*Hint: See Theorem V.1.1 in the lecture notes, and use that due to the condition $\Delta[M] = (\Delta M)^2$, we must have that $[M]_k - [M]_{k-1} = (M_k - M_{k-1})^2$.*

(d) Explicitly derive the predictable compensator $\langle M\rangle = \big(\langle M\rangle_k\big)_{k\in\mathbb{N}_0}$ of the process $M$ from (b).
*Hint: See the remark on page 79 in the lecture notes. Also use that if $M$ is a square-integrable martingale, then $\langle M\rangle$ is integrable.*

**Exercise 9.2** A *Poisson process* with parameter $\lambda > 0$ with respect to a probability measure $P$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a (real-valued) stochastic process $N = (N_t)_{t\geq 0}$ which is adapted to $\mathbb{F}$, $N_0 = 0$ $P$-a.s. and satisfies the following two properties:

(PP1) For $0 \leq s < t$, the *increment* $N_t - N_s$ is independent (under $P$) of $\mathcal{F}_s$ and is (under $P$) *Poisson-distributed* with parameter $\lambda(t - s)$, i.e.

$$P[N_t - N_s = k] = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k \in \mathbb{N}_0.$$

(PP2) $N$ is a *counting process* with jumps of size 1, i.e. for $P$-almost all $\omega \in \Omega$, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant, $\mathbb{N}_0$-valued, and increases by jumps of size 1.

Poisson processes form the cornerstone of *jump processes*, which are of importance in advanced financial modeling. Show that the following processes are $(P, \mathbb{F})$-martingales:

(a) $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$. This process is also called a *compensated Poisson process.*
*Hint: If $X \sim Poi(\lambda)$, then $E[X] = \lambda$.*

(b) $\widetilde{N}_t^2 - N_t$, $t \geq 0$, and $\widetilde{N}_t^2 - \lambda t$, $t \geq 0$. Use these results to derive $[\widetilde{N}]$ and $\langle \widetilde{N} \rangle$.
*Hint: If* $X \sim Poi(\lambda)$, *then* $\mathrm{Var}(X) = \lambda$.

(c) $S_t := e^{N_t \log(1+\sigma) - \lambda\sigma t}$, $t \geq 0$, where $\sigma > -1$. $S$ is also called a *geometric Poisson process.*

**Exercise 9.3** Let $(\Pi_n)_{n\in\mathbb{N}}$ be a sequence of refining partitions of $[a, b] \subseteq \mathbb{R}$ (in the sense that $\Pi_n \subseteq \Pi_{n+1}$ for all $n \in \mathbb{N}$) with $|\Pi_n| \to 0$ as $n \to \infty$. Let $p > 0$. We define for a function $f : \mathbb{R} \to \mathbb{R}$ its $p$-variation on $[a, b]$ along the sequence $(\Pi_n)_{n\in\mathbb{N}}$ as

$$V_p^{(a,b)}(f) := \lim_{n\to\infty} \sum_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})|^p,$$

assuming that the limit exists. Assume additionally that $f$ is continuous on $[a, b]$.

(a) Show that if $V_{p^*}^{(a,b)}(f)$ is finite and non-zero for some $p^* > 0$, then $V_p^{(a,b)}(f) = \infty$ for all $p < p^*$.
*Hint: Make sure to use the continuity of* $f$. *Use also that every function* $f : \mathbb{R} \to \mathbb{R}$ *that is continuous on a closed and bounded interval* $[a, b]$ *is also uniformly continuous on* $[a, b]$.

(b) Show that if $V_{p^*}^{(a,b)}(f)$ is finite and non-zero for some $p^* > 0$, then $V_p^{(a,b)}(f) = 0$ for all $p > p^*$.