ETH zürich

Mathematical Foundations for Finance

Exercise 4

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Arbitrage Opportunity

Definition 1 (Arbitrage opportunity)

An *arbitrage opportunity* is an admissible self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with zero initial wealth, with $V_T(\varphi) \geq 0$ P-a.s. and with $P[V_T(\varphi) > 0] > 0$. The financial market $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S^1)$ or shortly S is called *arbitrage-free* if there exist no arbitrage opportunities. Sometimes one also says that S satisfies (NA).

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- Self-financing and with zero initial investment at time $k = 0$ so that no external financing is needed.
- $V_T(\varphi) \geq 0$ P-a.s. so that we do not lose money P-a.s.
- $P[V_T(\varphi) > 0] > 0$ so that we stand a chance of making a gain.

Arbitrage Results in Finite Discrete Time

Lemma 2

Let $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S)$, or shortly S, with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$ be a financial market in finite discrete time. If there exists a probability measure $Q \approx P$ on $\mathcal{F}_T$ such that S is a Q-martingale, then the market S is arbitrage-free.

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- Note that it is if fact sufficient if there *exists an equivalent local martingale measure* (ELMM).
- This lemma actually holds for continuous time as well as infinite time horizon.
- The next big result shows that the converse holds as well.

Theorem 3 (Fundamental theorem of asset pricing)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S)$, or shortly S, with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ be a financial market in finite discrete time. Then S is arbitrage-free if and only if there exists an EMM for S.

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- In relation to the previous lemma, if S is a Q-martingale, then it is also a Q-local martingale.
- Unlike the previous lemma, this theorem does not in general hold in continuous time or infinite horizon.
- This helps us to express the rather complicated requirement of creating a model for a market that is free of arbitrage in terms of the simple notion of expectation.

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Corollary 4

The multiplicative multinomial model with parameters $y_1 < \ldots < y_m$ *and* r *is arbitrage-free if and only if* $y_1 < r < y_m$.

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The multiplicative binomial model with parameters $u > d$ *and* r *is arbitrage-free if and only if* $d < r < u$. *In that case the EMM for* S *is uniquely defined by*

$$q_u = Q[Y_k = 1 + u] = \frac{r - d}{u - d}.$$

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This makes intuitive sense. If $\widetilde{S}^1$ grew faster than $\widetilde{S}^0$ in all states of the world, then we would simply sell arbitrary amount of $\widetilde{S}^0$ and invest all the proceedings to $\widetilde{S}^1$ and we would have an arbitrage strategy.

Basic Financial Terms

This course is mainly about developing the theory required for pricing of financial derivatives. These will start to occur in the exercise sheets as well as in the lecture. While from mathematical perspective most of these instruments can solely be viewed as functions, it is good to understand why they take the forms that they take.

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- The underlying can vary a lot – interest rate, inflation, commodities, currencies, indices, stocks etc.
- In this course, we will deal exclusively with *equity derivatives*, for which the underlying are stocks, and more specifically with *options*.

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Definition 7 (European call option)

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- Since a rational investor would never exercise his or her option if the profit is negative, the payoffs of these options at time T can easily be seen to be $C(\omega) = \max\{0, \widetilde{S}_T(\omega) - K\}$ and $P(\omega) = \max\{0, K - \widetilde{S}_T(\omega)\}$.

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- The knowledge of the payoff still does not tell us too much about the price in between the inception of the contract and the maturity.
- It seems reasonable to set the price so that there is no arbitrage, which suggests that the price should be positive at all times.

Thank you for your attention!