Non-Life Insurance: Mathematics and Statistics

Solution sheet 13

Solution 13.1 Chain-Ladder Algorithm

(a) According to formula (9.5) of the lecture notes (version of March 20, 2019), the CL factor f_j can be estimated by

$$\widehat{f}_j^{\mathrm{CL}} = \frac{\sum_{i=1}^{I-j-1} C_{i,j+1}}{\sum_{i=1}^{I-j-1} C_{i,j}} = \sum_{i=1}^{I-j-1} \frac{C_{i,j}}{\sum_{n=1}^{I-j-1} C_{n,j}} \frac{C_{i,j+1}}{C_{i,j}},$$

for all $j = 0, \ldots, J-1$. Then, for all $i = 2, \ldots, I$ and $j = 1, \ldots, J$ with $i + j > I$, $C_{i,j}$ can be predicted by

$$\widehat{C}_{i,j}^{\mathrm{CL}} = C_{i,I-i} \prod_{k=I-i}^{j-1} \widehat{f}_k^{\mathrm{CL}},$$

see formula (9.6) of the lecture notes (version of March 20, 2019). In particular, for the prediction $\widehat{C}_{i,J}^{\mathrm{CL}}$ of the ultimate claim $C_{i,J}$ we have, for all $i = 2, \ldots, I$,

$$\widehat{C}_{i,J}^{\mathrm{CL}} = C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j^{\mathrm{CL}}. \tag{1}$$

The estimates $\widehat{f}_0^{\mathrm{CL}}, \ldots, \widehat{f}_{J-1}^{\mathrm{CL}}$ and the prediction of the lower triangle $\mathcal{D}_I^c$ are given in Table 1. We see that $\widehat{f}_0^{\mathrm{CL}} \approx 1.5$, while $\widehat{f}_j^{\mathrm{CL}}$ is close to 1 for all $j = 1, \ldots, J-1$, i.e. we observe a rather fast claims settlement in this example.

accident year i	development year j: 0	1	2	3	4	5	6	7	8	9
1										
2										10’663’318
3									10’646’884	10’662’008
4								9’734’574	9’744’764	9’758’606
5							9’837’277	9’847’906	9’858’214	9’872’218
6						10’005’044	10’056’528	10’067’393	10’077’931	10’092’247
7					9’419’776	9’485’469	9’534’279	9’544’580	9’554’571	9’568’143
8				8’445’057	8’570’389	8’630’159	8’674’568	8’683’940	8’693’030	8’705’378
9			8’243’496	8’432’051	8’557’190	8’616’868	8’661’208	8’670’566	8’679’642	8’691’971
10		8’470’989	9’129’696	9’338’521	9’477’113	9’543’206	9’592’313	9’602’676	9’612’728	9’626’383
$\widehat{f}_j^{\mathrm{CL}}$	1.493	1.078	1.023	1.015	1.007	1.005	1.001	1.001	1.001	

Table 1: Estimates $\widehat{f}_0^{\mathrm{CL}}, \ldots, \widehat{f}_{J-1}^{\mathrm{CL}}$ and prediction of the lower triangle $\mathcal{D}_I^c$.

(b) The CL reserves $\widehat{\mathcal{R}}_i^{\mathrm{CL}}$ at time $t = I$ are given by

$$\widehat{\mathcal{R}}_i^{\mathrm{CL}} = \widehat{C}_{i,J}^{\mathrm{CL}} - C_{i,I-i} = C_{i,I-i} \left(\prod_{j=I-i}^{J-1} \widehat{f}_j^{\mathrm{CL}} - 1 \right),$$

for all accident years $i = 2, \ldots, I$. Moreover, since $C_{1,J} = C_{1,I-1}$ is known, we have $\widehat{\mathcal{R}}_1^{\mathrm{CL}} = 0$.

Summarizing, we get the following CL reserves $\widehat{\mathcal{R}}_i^{\mathrm{CL}}$:

accident year i	1	2	3	4	5	6	7	8	9	10
CL reserves $\widehat{\mathcal{R}}_i^{\mathrm{CL}}$	0	15'126	26'257	34'538	85'302	156'494	286'121	449'167	1'043'242	3'950'815

Table 2: CL reserves $\widehat{\mathcal{R}}_i^{\mathrm{CL}}$ for all accident years $i = 1, \dots, I$.

By aggregating the CL reserves over all accident years, we get the CL predictor $\widehat{\mathcal{R}}^{\mathrm{CL}}$ for the outstanding loss liabilities of past exposure claims:

$$\widehat{\mathcal{R}}^{\mathrm{CL}} = \sum_{i=1}^{I} \widehat{\mathcal{R}}_i^{\mathrm{CL}} = 6'047'061.$$

Solution 13.2 Bornhuetter-Ferguson Algorithm

(a) Let $C_0 > 0$ be some initial value for development period $j = 0$. Then, for all $j = 0, \dots, J-1$ we define $\widehat{\beta}_j^{\mathrm{CL}}$ to be the proportion paid after the first j development periods according to the estimated CL pattern from Exercise 13.1. In particular, we calculate

$$\widehat{\beta}_0^{\mathrm{CL}} = \frac{C_0}{C_0 \prod_{l=0}^{J-1} \widehat{f}_l^{\mathrm{CL}}} = \frac{1}{\prod_{l=0}^{J-1} \widehat{f}_l^{\mathrm{CL}}} = \prod_{l=0}^{J-1} \frac{1}{\widehat{f}_l^{\mathrm{CL}}}$$

and

$$\widehat{\beta}_j^{\mathrm{CL}} = \frac{C_0 \prod_{l=0}^{j-1} \widehat{f}_l^{\mathrm{CL}}}{C_0 \prod_{l=0}^{J-1} \widehat{f}_l^{\mathrm{CL}}} = \frac{\prod_{l=0}^{j-1} \widehat{f}_l^{\mathrm{CL}}}{\prod_{l=0}^{J-1} \widehat{f}_l^{\mathrm{CL}}} = \prod_{l=j}^{J-1} \frac{1}{\widehat{f}_l^{\mathrm{CL}}},$$

for all $j = 1, \dots, J-1$. We get the following proportions:

development period j	0	1	2	3	4	5	6	7	8
proportion $\widehat{\beta}_j^{\mathrm{CL}}$ paid so far	0.590	0.880	0.948	0.970	0.984	0.991	0.996	0.998	0.999

Table 3: Proportions $\widehat{\beta}_j^{\mathrm{CL}}$ paid after the first j development periods according to the estimated CL pattern from Exercise 13.1.

According to formula (9.8) of the lecture notes (version of March 20, 2019), in the Bornhuetter-Ferguson method the ultimate claim $C_{i,J}$ is predicted by

$$\widehat{C}_{i,J}^{\mathrm{BF}} = C_{i,I-i} + \widehat{\mu}_i \left(1 - \widehat{\beta}_{I-i}^{\mathrm{CL}}\right),$$

for all accident years $i = 2, \dots, I$. Thus, the Bornhuetter-Ferguson reserves $\widehat{\mathcal{R}}_i^{\mathrm{BF}}$ are given by

$$\widehat{\mathcal{R}}_i^{\mathrm{BF}} = \widehat{C}_{i,J}^{\mathrm{BF}} - C_{i,I-i} = \widehat{\mu}_i \left(1 - \widehat{\beta}_{I-i}^{\mathrm{CL}}\right),$$

for all accident years $i = 2, \dots, I$. Moreover, since $C_{1,J} = C_{1,I-1}$ is known, we have $\widehat{\mathcal{R}}_1^{\mathrm{BF}} = 0$.

Summarizing, we get the following BF reserves $\widehat{\mathcal{R}}_i^{\mathrm{BF}}$:

accident year i	1	2	3	4	5	6	7	8	9	10
BF reserves $\widehat{\mathcal{R}}_i^{\mathrm{BF}}$	0	16'124	26'998	37'575	95'434	178'024	341'305	574'089	1'318'646	4'768'384

Table 4: BF reserves $\widehat{\mathcal{R}}_i^{\mathrm{BF}}$ for all accident years $i = 1, \ldots, I$.

By aggregating the BF reserves over all accident years, we get the BF predictor $\widehat{\mathcal{R}}^{\mathrm{BF}}$ for the outstanding loss liabilities of past exposure claims:

$$\widehat{\mathcal{R}}^{\mathrm{BF}} = \sum_{i=1}^{I} \widehat{\mathcal{R}}_i^{\mathrm{BF}} = 7'356'580.$$

(b) Note that for accident year $i = 1$ we have

$$\widehat{\mathcal{R}}_1^{\mathrm{CL}} = 0 = \widehat{\mathcal{R}}_1^{\mathrm{BF}}.$$

Now let $i = 2, \ldots, I$. Then, we observe that

$$\widehat{\mathcal{R}}_i^{\mathrm{CL}} < \widehat{\mathcal{R}}_i^{\mathrm{BF}}.$$

This can be explained as follows: Equation (1) can be rewritten as

$$\begin{aligned}\widehat{C}_{i,J}^{\mathrm{CL}} &= C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j^{\mathrm{CL}} = C_{i,I-i} + C_{i,I-i} \left(\prod_{j=I-i}^{J-1} \widehat{f}_j^{\mathrm{CL}} - 1 \right) \\ &= C_{i,I-i} + C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j^{\mathrm{CL}} \left(1 - \prod_{j=I-i}^{J-1} \frac{1}{\widehat{f}_j^{\mathrm{CL}}} \right) = C_{i,I-i} + \widehat{C}_{i,J}^{\mathrm{CL}} \left(1 - \widehat{\beta}_{I-i}^{\mathrm{CL}} \right).\end{aligned}$$

Comparing this to

$$\widehat{C}_{i,J}^{\mathrm{BF}} = C_{i,I-i} + \widehat{\mu}_i \left(1 - \widehat{\beta}_{I-i}^{\mathrm{CL}} \right)$$

and noting that for the prior information $\widehat{\mu}_i$ we have $\widehat{\mu}_i > \widehat{C}_{i,J}^{\mathrm{CL}}$, we immediately see that

$$\widehat{C}_{i,J}^{\mathrm{CL}} < \widehat{C}_{i,J}^{\mathrm{BF}},$$

which of course implies that

$$\widehat{\mathcal{R}}_i^{\mathrm{CL}} = \widehat{C}_{i,J}^{\mathrm{CL}} - C_{i,I-i} < \widehat{C}_{i,J}^{\mathrm{BF}} - C_{i,I-i} = \widehat{\mathcal{R}}_i^{\mathrm{BF}}.$$

We conclude that choosing a prior information $\widehat{\mu}_i$ which is bigger than the estimated CL ultimate $\widehat{C}_{i,J}^{\mathrm{CL}}$ leads to more conservative, i.e. higher reserves in the Bornhuetter-Ferguson method compared to the chain-ladder method.

Solution 13.3 Over-Dispersed Poisson Model

(a) According to Theorem 9.11 of the lecture notes (version of March 20, 2019), the MLEs $\widehat{\mu}_1^{\text{MLE}}, \dots, \widehat{\mu}_I^{\text{MLE}}$ and $\widehat{\gamma}_0^{\text{MLE}}, \dots, \widehat{\gamma}_J^{\text{MLE}}$ of $\mu_1, \dots, \mu_I$ and $\gamma_0, \dots, \gamma_J$ are given by

$$\widehat{\mu}_i^{\text{MLE}} = \widehat{C}_{i,J}^{\text{CL}} \qquad \text{and} \qquad \widehat{\gamma}_j^{\text{MLE}} = \left(1 - \frac{1}{\widehat{f}_{j-1}^{\text{CL}}}\right) \prod_{k=j}^{J-1} \frac{1}{\widehat{f}_k^{\text{CL}}},$$

for all $i = 1, \dots, I$ and $j = 1, \dots, J-1$, where $\widehat{C}_{i,J}^{\text{CL}}$ is the prediction of the ultimate claim $C_{i,J}$ and $\widehat{f}_j$ the estimated CL factor f_j from the chain-ladder model of Exercise 13.1. Moreover, we have

$$\widehat{\gamma}_0^{\text{MLE}} = \prod_{k=0}^{J-1} \frac{1}{\widehat{f}_k^{\text{CL}}} \qquad \text{and} \qquad \widehat{\gamma}_J^{\text{MLE}} = \left(1 - \frac{1}{\widehat{f}_{J-1}^{\text{CL}}}\right).$$

The values of the MLEs $\widehat{\mu}_1^{\text{MLE}}, \dots, \widehat{\mu}_I^{\text{MLE}}$ are given in Table 5, the values of the MLEs $\widehat{\gamma}_0^{\text{MLE}}, \dots, \widehat{\gamma}_J^{\text{MLE}}$ in Table 6.

accident year i	1	2	3	4	5
MLE $\widehat{\mu}_i^{\text{MLE}}$	11'148'124	10'663'318	10'662'008	9'758'606	9'872'218
accident year i	6	7	8	9	10
MLE $\widehat{\mu}_i^{\text{MLE}}$	10'092'247	9'568'143	8'705'378	8'691'971	9'626'383

Table 5: Values of the MLEs $\widehat{\mu}_1^{\text{MLE}}, \dots, \widehat{\mu}_I^{\text{MLE}}$.

development year j	0	1	2	3	4	5	6	7	8	9
MLE $\widehat{\gamma}_j^{\text{MLE}}$	0.590	0.290	0.068	0.022	0.014	0.007	0.005	0.001	0.001	0.001

Table 6: Values of the MLEs $\widehat{\gamma}_0^{\text{MLE}}, \dots, \widehat{\gamma}_J^{\text{MLE}}$.

(b) According to Theorem 9.11 of the lecture notes (version of March 20, 2019), the ODP reserves $\widehat{\mathcal{R}}_i^{\text{ODP}}$ are given by

$$\widehat{\mathcal{R}}_i^{\text{ODP}} = \widehat{\mu}_i^{\text{MLE}} \sum_{j=I-i+1}^{J} \widehat{\gamma}_j^{\text{MLE}},$$

for all accident years $i = 2, \dots, I$. Moreover, since $C_{1,J} = C_{1,I-1}$ is known, we have $\widehat{\mathcal{R}}_1^{\text{ODP}} = 0$. Summarizing, we get the following ODP reserves $\widehat{\mathcal{R}}_i^{\text{ODP}}$:

accident year i	1	2	3	4	5	6	7	8	9	10
ODP reserves $\widehat{\mathcal{R}}_i^{\text{ODP}}$	0	15'126	26'257	34'538	85'302	156'494	286'121	449'167	1'043'242	3'950'815

Table 7: ODP reserves $\widehat{\mathcal{R}}_i^{\text{ODP}}$ for all accident years $i = 1, \dots, I$.

We observe that $\widehat{\mathcal{R}}_i^{\text{ODP}} = \widehat{\mathcal{R}}_i^{\text{CL}}$ for all accident years $i = 1, \dots, I$, where $\widehat{\mathcal{R}}_i^{\text{CL}}$ are the CL reserves from Exercise 13.1. As a matter of fact, this observation holds true in general, see Theorem 9.11 of the lecture notes (version of March 20, 2019). By aggregating the ODP reserves over all accident years, we get the ODP predictor $\widehat{\mathcal{R}}^{\text{ODP}}$ for the outstanding loss liabilities of past exposure claims (which is equal to the CL predictor $\widehat{\mathcal{R}}^{\text{CL}}$):

$$\widehat{\mathcal{R}}^{\text{ODP}} = \sum_{i=1}^{I} \widehat{\mathcal{R}}_i^{\text{ODP}} = 6'047'061 = \sum_{i=1}^{I} \widehat{\mathcal{R}}_i^{\text{CL}} = \widehat{\mathcal{R}}^{\text{CL}}.$$

(c) As the ODP model belongs to the family of GLM models, we can calculate the ODP reserves also using the GLM machinery. In particular, we work with the two risk characteristics accident year i, with parameters $\beta_{1,1}, \dots, \beta_{1,I}$, and development year j, with parameters $\beta_{2,0}, \dots, \beta_{2,J}$, where $\beta_{1,i}$ corresponds to accident year i and $\beta_{2,j}$ to development year j. Compared to the parametrization on the exercise sheet, in order to apply GLM techniques, we use the following re-parametrization. We assume that all $X_{i,j}$ are independent with

$$\frac{X_{i,j}}{\phi} \sim \mathrm{Poi}(\lambda_{i,j}/\phi),$$

for all risk classes $(i,j), 1 \le i \le I, 0 \le j \le J$, where $\lambda_{i,j}$ denotes the mean parameter. Note that we work with volumes which are constantly equal to 1. Moreover, in a Poisson GLM model we would set $\phi = 1$. Here we assume a general dispersion parameter $\phi > 0$. We have

$$\mathbb{E}[X_{i,j}] = \phi\, \mathbb{E}\left[\frac{X_{i,j}}{\phi}\right] = \phi \frac{\lambda_{i,j}}{\phi} = \lambda_{i,j},$$

and we model

$$g(\lambda_{i,j}) = g(\mathbb{E}[X_{i,j}]) = \beta_0 + \beta_{1,i} + \beta_{2,j},$$

where $\beta_0 \in \mathbb{R}$ and where we use the log-link function, i.e. $g(\cdot) = \log(\cdot)$. In order to get a unique solution, we set $\beta_{1,1} = \beta_{2,0} = 0$. We refer to Listing 1 for the application of this over-dispersed Poisson GLM model in R.

Listing 1: R code for Exercise 13.3 (c).

```
### Load the required packages
library(readxl)
library(plyr)

### Download the data from the link indicated on the exercise sheet
### Store the data under the name "Exercise13Data.xls" in the same folder as this R code
### Load the data
data <- read_excel("Exercise13Data.xls", sheet="Data_1", range="B22:K31", col_names=FALSE)

### ODP as GLM Model
data2 <- as.data.frame(data)
data2[,2:10] <- data2[,2:10]-data2[,1:9]
data2 <- stack(data2, select=c("X__1","X__2","X__3","X__4","X__5","X__6","X__7","X__8","X__9",
                               "X__10"))
data2[,2] <- rep(1:10)
data2[,3] <- rep(0:9,each=10)
colnames(data2)[2:3] <- c("AY","DY")
data2$AY <- as.factor(data2$AY)
data2$DY <- as.factor(data2$DY)
lower.ind <- is.na(data2[,1])
upper <- data2[is.na(data2[,1])==FALSE,]
lower <- data2[is.na(data2[,1]),]
ODP <- glm(values ~ AY+DY, data=upper, family=quasipoisson())
lower[,1] <- predict(ODP, newdata=lower, "response")
ODP.GLM.reserves <- rep(0,10)
ODP.GLM.reserves[1] <- 0
ODP.GLM.reserves[2:10] <- ddply(lower, .(AY), summarise, reserves=sum(values))[,2]
round(ODP.GLM.reserves)

### MLEs for the accident years
exp(c(0,ODP$coefficients[2:10])+ODP$coefficients[1])*sum(exp(c(0,ODP$coefficients[11:19])))

### MLEs for the development years
round(exp(c(0,ODP$coefficients[11:19]))/sum(exp(c(0,ODP$coefficients[11:19]))),3)
```

Running the R code of Listing 1, we can confirm that the ODP GLM model leads to the same reserves as the CL method. In order to check the MLE parameters of Tables 5 and 6, we have to go back to the parametrization used on the exercise sheet. We write

$\widehat{\beta}_0^{\mathrm{MLE}}, \widehat{\beta}_{1,1}^{\mathrm{MLE}}, \ldots, \widehat{\beta}_{1,I}^{\mathrm{MLE}}, \widehat{\beta}_{2,0}^{\mathrm{MLE}}, \ldots, \widehat{\beta}_{2,J}^{\mathrm{MLE}}$ for the resulting MLEs of the ODP GLM model. By setting

$$\widetilde{\mu}_i \;=\; \exp\left\{\widehat{\beta}_0^{\mathrm{MLE}} + \widehat{\beta}_{1,i}^{\mathrm{MLE}}\right\} \sum_{k=0}^{J} \exp\left\{\widehat{\beta}_{2,k}^{\mathrm{MLE}}\right\},$$

for all $i = 1, \ldots, I$, and

$$\widetilde{\gamma}_j \;=\; \frac{\exp\left\{\widehat{\beta}_{2,j}^{\mathrm{MLE}}\right\}}{\sum_{k=0}^{J} \exp\left\{\widehat{\beta}_{2,k}^{\mathrm{MLE}}\right\}},$$

for all $j = 0, \ldots, J$, we get

$$\begin{aligned}
\widehat{\lambda}_{i,j}^{\mathrm{MLE}} &= \exp\left\{\widehat{\beta}_0^{\mathrm{MLE}} + \widehat{\beta}_{1,i}^{\mathrm{MLE}} + \widehat{\beta}_{2,j}^{\mathrm{MLE}}\right\} \\
&= \exp\left\{\widehat{\beta}_0^{\mathrm{MLE}} + \widehat{\beta}_{1,i}^{\mathrm{MLE}}\right\} \sum_{k=0}^{J} \exp\left\{\widehat{\beta}_{2,k}^{\mathrm{MLE}}\right\} \frac{\exp\left\{\widehat{\beta}_{2,j}^{\mathrm{MLE}}\right\}}{\sum_{k=0}^{J} \exp\left\{\widehat{\beta}_{2,k}^{\mathrm{MLE}}\right\}} \\
&= \widetilde{\mu}_i \widetilde{\gamma}_i.
\end{aligned}$$

In particular, we get back to the parametrization used on the exercise sheet. The values of $\widetilde{\mu}_i$, $i = 1, \ldots, I$ and $\widetilde{\gamma}_j$, $j = 0, \ldots, J$, are calculated on lines 26 and 29 of Listing 1. We can confirm that we get the same values as in Tables 5 and 6.

Solution 13.4 Mack's Formula and Merz-Wüthrich (MW) Formula

(a) The R code used in this exercise is provided in Listing 2. We get the following results:

accident year i	CL reserves $\widehat{\mathcal{R}}_i^{CL}$	$\sqrt{\text{total msep}}$ (Mack)	in % of the reserves	$\sqrt{\text{CDR msep}}$ (MW)	in % of the $\sqrt{\text{total msep}}$
1	0	–	–	–	–
2	15'126	267	1.8 %	267	100 %
3	26'257	914	3.5 %	884	97 %
4	34'538	3'058	8.9 %	2'948	96 %
5	85'302	7'628	8.9 %	7'018	92 %
6	156'494	33'341	21.3 %	32'470	97 %
7	286'121	73'467	25.7 %	66'178	90 %
8	449'167	85'398	19.0 %	50'296	59 %
9	1'043'242	134'337	12.9 %	104'311	78 %
10	3'950'815	410'817	10.4 %	385'773	94 %
total	6'047'061	462'960	7.7 %	420'220	91 %

Table 8: CL reserves $\widehat{\mathcal{R}}_i^{CL}$, Mack's square-rooted conditional mean square errors of prediction and MW's square-rooted conditional mean square errors of prediction for all accident years $i = 1, \ldots, I$.

(b) Mack's square-rooted conditional mean square errors of prediction give us confidence bounds around the CL reserves. We see that for the total claims reserves the one standard deviation confidence bounds are 7.7%. The biggest uncertainties can be found for accident years 6, 7 and 8, where the one standard deviation confidence bounds are roughly 20% or even higher.

(c) MW's square-rooted conditional mean square errors of prediction measure the contribution of the next accounting year to the total (run-off) uncertainty given by Mack's square-rooted conditional mean square errors of prediction. For aggregated accident years, we see that 91% of the total uncertainty is due to the next accounting year. This high value can be explained by the fast claims settlement already discovered in Exercise 13.1, (a).

Listing 2: R code for Exercise 13.4 (a).

```
### Load the required packages
library(readxl)
library(ChainLadder)

### Download the data from the link indicated on the exercise sheet
### Store the data under the name "Exercise13Data.xls" in the same folder as this R code
### Load the data
data <- read_excel("Exercise13Data.xls", sheet="Data_1", range="B22:K31", col_names=FALSE)

### Bring the data in the appropriate triangular form and label the axes
tri <- as.triangle(as.matrix(data))
dimnames(tri)=list(origin=1:nrow(tri),dev=1:ncol(tri))

### Calculate the CL reserves and the corresponding mseps
M <- MackChainLadder(tri, est.sigma="Mack")

### CL reserves and Mack's square-rooted mseps (including illustrations)
M
plot(M)
plot(M, lattice=TRUE)

### CL reserves, MW's square-rooted mseps and Mack's square-rooted mseps
CDR(M)

### Mack's square-rooted mseps in % of the reserves
round(CDR(M)[,3]/CDR(M)[,1],3)*100

### MW's square-rooted mseps in % of Mack's square-rooted mseps
round(CDR(M)[,2]/CDR(M)[,3],2)*100

### Full uncertainty picture
CDR(M, dev="all")
```