Non-Life Insurance: Mathematics and Statistics

Solution sheet 3

Solution 3.1 No-Claims Bonus

(a) We define the following events:

$$\begin{aligned} A &= \{\text{“no claims in the last six years”}\}, \\ B &= \{\text{“no claims in the last three years but at least one claim in the last six years”}\}, \\ C &= \{\text{“at least one claim in the last three years”}\}. \end{aligned}$$

Note that since the events A, B and C are disjoint and cover all possible outcomes, we have

$$\mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] = 1,$$

i.e. it is sufficient to calculate two out of the three probabilities. Since the calculation of $\mathbb{P}[B]$ is slightly more involved, we will look at $\mathbb{P}[A]$ and $\mathbb{P}[C]$. Let $N_1, \ldots, N_6$ be the number of claims of the last six years of our considered car driver, where N_6 corresponds to the most recent year. By assumption, $N_1, \ldots, N_6$ are i.id. Poisson random variables with frequency parameter $\lambda = 0.2$. Therefore, we can calculate

$$\mathbb{P}[A] = \mathbb{P}\left[N_1 = 0, \ldots, N_6 = 0\right] = \prod_{i=1}^{6} \mathbb{P}\left[N_i = 0\right] = \prod_{i=1}^{6} \exp\{-\lambda\} = \exp\{-6\lambda\} = \exp\{-1.2\}$$

and, similarly,

$$\mathbb{P}[C] = 1 - \mathbb{P}[C^c] = 1 - \mathbb{P}\left[N_4 = 0, N_5 = 0, N_6 = 0\right] = 1 - \exp\{-3\lambda\} = 1 - \exp\{-0.6\}.$$

For the event B we get

$$\mathbb{P}[B] = 1 - \mathbb{P}[A] - \mathbb{P}[C] = 1 - \exp\{-1.2\} - (1 - \exp\{-0.6\}) = \exp\{-0.6\} - \exp\{-1.2\}.$$

Thus, the expected proportion q of the premium that is still paid after the grant of the no-claims bonus is given by

$$\begin{aligned} q &= \mathbb{E}[0.8 \cdot 1_A + 0.9 \cdot 1_B + 1 \cdot 1_C] = 0.8 \cdot \mathbb{P}[A] + 0.9 \cdot \mathbb{P}[B] + 1 \cdot \mathbb{P}[C] \\ &= 0.8 \cdot \exp\{-1.2\} + 0.9 \cdot (\exp\{-0.6\} - \exp\{-1.2\}) + 1 - \exp\{-0.6\} \\ &\approx 0.915. \end{aligned}$$

If s denotes the surcharge on the premium, then it has to satisfy the equation

$$q\,(1+s) \cdot \text{premium} = \text{premium},$$

which leads to

$$s = \frac{1}{q} - 1.$$

We conclude that the surcharge on the premium is given by approximately 9.3%.

(b) We use the same notation as in (a). Since this time the calculation of $\mathbb{P}[B]$ is considerably more involved, we again look at $\mathbb{P}[A]$ and $\mathbb{P}[C]$. By assumption, conditionally given Θ, $N_1, \ldots, N_6$ are i.i.d. Poisson random variables with frequency parameter $\Theta\lambda$, where $\lambda = 0.2$. Therefore, we can calculate

$$\mathbb{P}[A] \,=\, \mathbb{P}\left[N_1 = 0, \ldots, N_6 = 0\right] \,=\, \mathbb{E}\left[\mathbb{P}\left[N_1 = 0, \ldots, N_6 = 0 | \Theta\right]\right] \,=\, \mathbb{E}\left[\prod_{i=1}^{6} \mathbb{P}\left[N_i = 0 | \Theta\right]\right]$$
$$= \mathbb{E}\left[\prod_{i=1}^{6} \exp\{-\Theta\lambda\}\right] \,=\, \mathbb{E}\left[\exp\{-6\Theta\lambda\}\right] \,=\, M_\Theta(-6\lambda),$$

where M_Θ denotes the moment generating function of Θ. Since Θ has an exponential distribution with parameter $c = 1$, M_Θ is given by

$$M_\Theta(r) \,=\, \frac{1}{1-r},$$

for all $r < 1$, see Exercise 1.2, which leads to

$$\mathbb{P}[A] \,=\, \frac{1}{1+6\lambda} \,=\, \frac{1}{2.2}.$$

Similarly, we get

$$\mathbb{P}[C] \,=\, 1 - \mathbb{P}[C^c] \,=\, 1 - \mathbb{P}\left[N_4 = 0, N_5 = 0, N_6 = 0\right] \,=\, 1 - \frac{1}{1+3\lambda} \,=\, 1 - \frac{1}{1.6} \,=\, \frac{0.6}{1.6}.$$

For the event B we get

$$\mathbb{P}[B] \,=\, 1 - \mathbb{P}[A] - \mathbb{P}[C] \,=\, 1 - \frac{1}{2.2} - \frac{0.6}{1.6} \,=\, \frac{1}{1.6} - \frac{1}{2.2}.$$

Thus, the expected proportion q of the premium that is still paid after the grant of the no-claims bonus is given by

$$q \,=\, 0.8 \cdot \mathbb{P}[A] + 0.9 \cdot \mathbb{P}[B] + 1 \cdot \mathbb{P}[C] \,=\, 0.8 \cdot \frac{1}{2.2} + 0.9 \cdot \left(\frac{1}{1.6} - \frac{1}{2.2}\right) + \frac{0.6}{1.6} \,\approx\, 0.892.$$

We conclude that the surcharge s on the premium is given by

$$s \,=\, \frac{1}{q} - 1 \,\approx\, 12.1\%,$$

which is considerably bigger than in (a). The reason is that in (b) we introduce dependence between the claim counts of the individual years of the considered car driver. This increases the probability of having no claims in the last six years, and decreases the expected proportion q of the premium that is still paid after the grant of the no-claims bonus.

Solution 3.2 Compound Poisson Distribution

(a) Since $S \sim \text{CompPoi}(\lambda v, G)$, we can write S as

$$S \,=\, \sum_{i=1}^{N} Y_i,$$

where $N \sim \text{Poi}(\lambda v)$, $Y_1, Y_2, \ldots$ are i.i.d. with distribution function G and N and $Y_1, Y_2, \ldots$ are independent. Now we can define S_{sc}, S_{mc} and S_{lc} as

$$S_{\text{sc}} \,=\, \sum_{i=1}^{N} Y_i 1_{\{Y_i \leq 1\text{'}000\}}, \;\; S_{\text{mc}} \,=\, \sum_{i=1}^{N} Y_i 1_{\{1\text{'}000 < Y_i \leq 1\text{'}000\text{'}000\}} \text{ and } S_{\text{lc}} \,=\, \sum_{i=1}^{N} Y_i 1_{\{Y_i > 1\text{'}000\text{'}000\}}.$$

(b) Note that according to Table 2 given on the exercise sheet, we have

$$\begin{aligned}
\mathbb{P}[Y_1 \leq 1'000] &= \mathbb{P}[Y_1 = 100] + \mathbb{P}[Y_1 = 300] + \mathbb{P}[Y_1 = 500] = \frac{3}{20} + \frac{4}{20} + \frac{3}{20} = \frac{1}{2}, \\
\mathbb{P}[1'000 < Y_1 \leq 1'000'000] &= \mathbb{P}[Y_1 = 6'000] + \mathbb{P}[Y_1 = 100'000] + \mathbb{P}[Y_1 = 500'000] \\
&= \frac{2}{15} + \frac{2}{15} + \frac{1}{15} = \frac{1}{3} \quad \text{and} \\
\mathbb{P}[Y_1 > 1'000'000] &= 1 - \mathbb{P}[Y_1 \leq 1'000'000] = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
\end{aligned}$$

Thus, using Theorem 2.14 (disjoint decomposition of compound Poisson distributions) of the lecture notes (version of March 20, 2019), we get

$$S_{\mathrm{sc}} \sim \mathrm{CompPoi}\left(\frac{\lambda v}{2}, G_{\mathrm{sc}}\right), \; S_{\mathrm{mc}} \sim \mathrm{CompPoi}\left(\frac{\lambda v}{3}, G_{\mathrm{mc}}\right) \text{ and } S_{\mathrm{lc}} \sim \mathrm{CompPoi}\left(\frac{\lambda v}{6}, G_{\mathrm{lc}}\right),$$

where

$$\begin{aligned}
G_{\mathrm{sc}}(y) &= \mathbb{P}[Y_1 \leq y | Y_1 \leq 1'000], \\
G_{\mathrm{mc}}(y) &= \mathbb{P}[Y_1 \leq y | 1'000 < Y_1 \leq 1'000'000] \quad \text{and} \\
G_{\mathrm{lc}}(y) &= \mathbb{P}[Y_1 \leq y | Y_1 > 1'000'000],
\end{aligned}$$

for $y \in \mathbb{R}$. In particular, for a random variable Y_{sc} having distribution function G_{sc}, we have

$$\begin{aligned}
\mathbb{P}[Y_{\mathrm{sc}} = 100] &= \frac{\mathbb{P}[Y_1 = 100]}{\mathbb{P}[Y_1 \leq 1'000]} = \frac{3/20}{1/2} = \frac{3}{10}, \\
\mathbb{P}[Y_{\mathrm{sc}} = 300] &= \frac{\mathbb{P}[Y_1 = 300]}{\mathbb{P}[Y_1 \leq 1'000]} = \frac{4/20}{1/2} = \frac{4}{10} \quad \text{and} \\
\mathbb{P}[Y_{\mathrm{sc}} = 500] &= \frac{\mathbb{P}[Y_1 = 500]}{\mathbb{P}[Y_1 \leq 1'000]} = \frac{3/20}{1/2} = \frac{3}{10}.
\end{aligned}$$

Analogously, for random variables Y_{mc} and Y_{lc} having distribution functions G_{mc} and G_{lc}, respectively, we get

$$\mathbb{P}[Y_{\mathrm{mc}} = 6'000] = \frac{2}{5}, \quad \mathbb{P}[Y_{\mathrm{mc}} = 100'000] = \frac{2}{5} \quad \text{and} \quad \mathbb{P}[Y_{\mathrm{mc}} = 500'000] = \frac{1}{5},$$

as well as

$$\mathbb{P}[Y_{\mathrm{lc}} = 2'000'000] = \frac{1}{2}, \quad \mathbb{P}[Y_{\mathrm{lc}} = 5'000'000] = \frac{1}{4} \quad \text{and} \quad \mathbb{P}[Y_{\mathrm{lc}} = 10'000'000] = \frac{1}{4}.$$

(c) According to Theorem 2.14 of the lecture notes (version of March 20, 2019), S_{sc}, S_{mc} and S_{lc} are independent.

(d) In order to find $\mathbb{E}[S_{\mathrm{sc}}]$, we need $\mathbb{E}[Y_{\mathrm{sc}}]$, which can be calculated as

$$\mathbb{E}[Y_{\mathrm{sc}}] = 100 \cdot \mathbb{P}[Y_{\mathrm{sc}} = 100] + 300 \cdot \mathbb{P}[Y_{\mathrm{sc}} = 300] + 500 \cdot \mathbb{P}[Y_{\mathrm{sc}} = 500] = \frac{300}{10} + \frac{1200}{10} + \frac{1500}{10} = 300.$$

Now we can apply Proposition 2.11 of the lecture notes (version of March 20, 2019) to get

$$\mathbb{E}[S_{\mathrm{sc}}] = \frac{\lambda v}{2} \mathbb{E}[Y_{\mathrm{sc}}] = 0.3 \cdot 300 = 90.$$

Similarly, we get

$$\mathbb{E}[Y_{\mathrm{mc}}] = 142'400 \quad \text{and} \quad \mathbb{E}[Y_{\mathrm{lc}}] = 4'750'000.$$

Thus, we find

$$\mathbb{E}[S_{\mathrm{mc}}] = \frac{\lambda v}{3}\mathbb{E}[Y_{\mathrm{mc}}] = 28'480 \quad \text{and} \quad \mathbb{E}[S_{\mathrm{lc}}] = \frac{\lambda v}{6}\mathbb{E}[Y_{\mathrm{lc}}] = 475'000.$$

Since $S = S_{\mathrm{sc}} + S_{\mathrm{mc}} + S_{\mathrm{lc}}$, we get

$$\mathbb{E}[S] = \mathbb{E}[S_{\mathrm{sc}}] + \mathbb{E}[S_{\mathrm{mc}}] + \mathbb{E}[S_{\mathrm{lc}}] = 503'570.$$

In order to find $\mathrm{Var}(S_{\mathrm{sc}})$, we need $\mathbb{E}[Y_{\mathrm{sc}}^2]$, which can be calculated as

$$\begin{aligned}\mathbb{E}[Y_{\mathrm{sc}}^2] &= 100^2 \cdot \mathbb{P}[Y_{\mathrm{sc}} = 100] + 300^2 \cdot \mathbb{P}[Y_{\mathrm{sc}} = 300] + 500^2 \cdot \mathbb{P}[Y_{\mathrm{sc}} = 500] \\ &= \frac{30'000}{10} + \frac{360'000}{10} + \frac{750'000}{10} = 114'000.\end{aligned}$$

Now we can apply Proposition 2.11 of the lecture notes (version of March 20, 2019) to get

$$\mathrm{Var}(S_{\mathrm{sc}}) = \frac{\lambda v}{2}\mathbb{E}[Y_{\mathrm{sc}}^2] = 0.3 \cdot 114'000 = 34'200.$$

Similarly, we get

$$\mathbb{E}[Y_{\mathrm{mc}}^2] = 54'014'400'000 \quad \text{and} \quad \mathbb{E}[Y_{\mathrm{lc}}^2] = 33'250'000'000'000.$$

Thus, we find

$$\mathrm{Var}(S_{\mathrm{mc}}) = \frac{\lambda v}{3}\mathbb{E}[Y_{\mathrm{mc}}^2] = 10'802'880'000 \quad \text{and} \quad \mathrm{Var}(S_{\mathrm{lc}}) = \frac{\lambda v}{6}\mathbb{E}[Y_{\mathrm{lc}}^2] = 3'325'000'000'000.$$

Since $S = S_{\mathrm{sc}} + S_{\mathrm{mc}} + S_{\mathrm{lc}}$, and S_{sc}, S_{mc} and S_{lc} are independent, we get

$$\sqrt{\mathrm{Var}(S)} = \sqrt{\mathrm{Var}(S_{\mathrm{sc}}) + \mathrm{Var}(S_{\mathrm{mc}}) + \mathrm{Var}(S_{\mathrm{lc}})} = \sqrt{3'335'802'914'200} \approx 1'826'418.$$

(e) First, we define the random variable N_{lc} as

$$N_{\mathrm{lc}} \sim \mathrm{Poi}\left(\frac{\lambda v}{6}\right).$$

The probability that the total claim in the large claims layer exceeds 5 million can be calculated by looking at the complement, i.e. at the probability that the total claim in the large claims layer does not exceed 5 million. Since the smallest claim size for a claim in the large claims layer is given by 2'000'000, with three claims in the large claims layer we already exceed 5 million with probability one. Thus, it is enough to consider only up to two claims. We get

$$\begin{aligned}&\mathbb{P}\left[S_{\mathrm{lc}} \leq 5'000'000\right] \\ &\quad = \mathbb{P}[N_{\mathrm{lc}} = 0] + \mathbb{P}[N_{\mathrm{lc}} = 1]\mathbb{P}[Y_{\mathrm{lc}} \leq 5'000'000] + \mathbb{P}[N_{\mathrm{lc}} = 2]\mathbb{P}[Y_{\mathrm{lc}} = 2'000'000]^2 \\ &\quad = \exp\left\{-\frac{\lambda v}{6}\right\} + \exp\left\{-\frac{\lambda v}{6}\right\}\frac{\lambda v}{6}\left(\frac{1}{2} + \frac{1}{4}\right) + \exp\left\{-\frac{\lambda v}{6}\right\}\left(\frac{\lambda v}{6}\right)^2 \frac{1}{2}\left(\frac{1}{2}\right)^2 \\ &\quad = \exp\left\{-0.1\right\}(1 + 0.075 + 0.00125) \\ &\quad \approx 97.4\%.\end{aligned}$$

We can conclude that

$$\mathbb{P}\left[S_{\mathrm{lc}} > 5'000'000\right] = 1 - \mathbb{P}\left[S_{\mathrm{lc}} \leq 5'000'000\right] \approx 2.6\%.$$

Solution 3.3 Compound Distribution

We show that the moment generating function M_S of S is equal to the moment generating function of an exponential distribution with parameter λp. According to Proposition 2.2 of the lecture notes (version of March 20, 2019), M_S is given by (wherever it exists)

$$M_S(r) \;=\; M_N[\log M_{Y_1}(r)],$$

where M_N and M_{Y_1} are the moment generating functions of N and Y_1, respectively. As $S \geq 0$ almost surely, $M_S(r)$ exists at least for all $r < 0$. In Exercises 1.1 and 1.2 we have seen that

$$M_N(r) \;=\; \frac{p \exp\{r\}}{1-(1-p)\exp\{r\}},$$

for all $r < -\log(1-p)$, and that

$$\log M_{Y_1}(r) \;=\; \log\left(\frac{\lambda}{\lambda - r}\right),$$

for all $r < \lambda$. Thus, we get

$$M_S(r) \;=\; M_N\left[\log\left(\frac{\lambda}{\lambda-r}\right)\right] \;=\; \frac{p\exp\left\{\log\left(\frac{\lambda}{\lambda-r}\right)\right\}}{1-(1-p)\exp\left\{\log\left(\frac{\lambda}{\lambda-r}\right)\right\}} \;=\; \frac{\lambda p}{\lambda - r - \lambda(1-p)} \;=\; \frac{\lambda p}{\lambda p - r}.$$

With Lemma 1.3 of the lecture notes (version of March 20, 2019), we conclude that S has indeed an exponential distribution with parameter λp. We remark that for this compound model the corresponding distribution function can be given in closed form. However, usually this is not possible. Therefore, we will consider other methods for the calculation of the distribution function of S in Chapter 4 of the lecture notes (version of March 20, 2019).

Solution 3.4 Compound Binomial Distribution

(a) Let $\tilde{S} \sim \mathrm{CompBinom}(\tilde{v}, \tilde{p}, \tilde{G})$ with the random variable $\tilde{Y}_1$ having distribution function $\tilde{G}$ and moment generating function $M_{\tilde{Y}_1}$. Then, by Proposition 2.6 of the lecture notes (version of March 20, 2019), the moment generating function $M_{\tilde{S}}$ of $\tilde{S}$ is given by

$$M_{\tilde{S}}(r) \;=\; \left(\tilde{p} M_{\tilde{Y}_1}(r) + 1 - \tilde{p}\right)^{\tilde{v}},$$

for all $r \in \mathbb{R}$ for which $M_{\tilde{Y}_1}$ is defined. We calculate the moment generating function $M_{S_{\mathrm{lc}}}$ of S_{lc} and show that it is exactly of the form given above. Let $r \in \mathbb{R}$ such that $M_{S_{\mathrm{lc}}}(r)$ exists. Note that since $S_{\mathrm{lc}} \geq 0$ almost surely, its moment generating function is defined at least for all $r < 0$. We have

$$\begin{aligned} M_{S_{\mathrm{lc}}}(r) \;&=\; \mathbb{E}\left[\exp\{rS_{\mathrm{lc}}\}\right] \;=\; \mathbb{E}\left[\exp\left\{r\sum_{i=1}^N Y_i\, 1_{\{Y_i > M\}}\right\}\right] \;=\; \mathbb{E}\left[\prod_{i=1}^N \exp\left\{rY_i\, 1_{\{Y_i>M\}}\right\}\right] \\ &=\; \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^N \exp\left\{rY_i\, 1_{\{Y_i>M\}}\right\}\middle| N\right]\right] \;=\; \mathbb{E}\left[\prod_{i=1}^N \mathbb{E}\left[\exp\left\{rY_i\, 1_{\{Y_i>M\}}\right\}\right]\right], \end{aligned}$$

where in the fourth equality we used the tower property of conditional expectation and in the fifth equality the independence between N and Y_i. For the inner expectation we get

$$\begin{aligned} \mathbb{E}\left[\exp\left\{rY_i\, 1_{\{Y_i>M\}}\right\}\right] \;&=\; \mathbb{E}\left[\exp\{rY_i\}\cdot 1_{\{Y_i>M\}} + 1_{\{Y_i \leq M\}}\right] \\ &=\; \mathbb{E}\left[\exp\{rY_i\}\,|Y_i > M\right]\mathbb{P}[Y_i > M] + \mathbb{P}[Y_i \leq M] \\ &=\; \mathbb{E}\left[\exp\{rY_i\}\,|Y_i > M\right][1-G(M)] + G(M). \end{aligned}$$

Note that the distribution function of the random variable $Y_i | Y_i > M$ is G_{lc}. Thus, we can write

$$\mathbb{E}\left[\exp\left\{rY_i \, 1_{\{Y_i>M\}}\right\}\right] = M_{Y_1|Y_1>M}(r)[1-G(M)] + G(M).$$

Hence, we get

$$\begin{aligned} M_{S_{\mathrm{lc}}}(r) &= \mathbb{E}\left[\prod_{i=1}^{N}\left(M_{Y_1|Y_1>M}(r)[1-G(M)] + G(M)\right)\right] \\ &= \mathbb{E}\left[\left(M_{Y_1|Y_1>M}(r)[1-G(M)] + G(M)\right)^N\right] \\ &= \mathbb{E}\left[\exp\left\{N \log\left(M_{Y_1|Y_1>M}(r)[1-G(M)] + G(M)\right)\right\}\right] \\ &= M_N(\rho), \end{aligned}$$

where M_N is the moment generating function of N and

$$\rho = \log\left(M_{Y_1|Y_1>M}(r)[1-G(M)] + G(M)\right).$$

Since we have $N \sim \mathrm{Binom}(v, p)$, $M_N(r)$ is given by

$$M_N(r) = (p \exp\{r\} + 1 - p)^v.$$

Therefore, we get

$$\begin{aligned} M_{S_{\mathrm{lc}}}(r) &= \left[p\left(M_{Y_1|Y_1>M}(r)[1-G(M)] + G(M)\right) + 1 - p\right]^v \\ &= \left(p[1-G(M)]M_{Y_1|Y_1>M}(r) + 1 - p[1-G(M)]\right)^v. \end{aligned}$$

Applying Lemma 1.3 of the lecture notes (version of March 20, 2019), we conclude that $S_{\mathrm{lc}} \sim \mathrm{CompBinom}(\tilde{v}, \tilde{p}, \tilde{G})$ with $\tilde{v} = v$, $\tilde{p} = p[1-G(M)]$ and $\tilde{G} = G_{\mathrm{lc}}$.

(b) In (a) we showed that the number of claims of the compound distribution S_{lc} has a binomial distribution with parameters v and $p[1-G(M)] > 0$. In particular, there is a positive probability that we have v claims with $Y_i > M$. Now suppose that $S_{\mathrm{sc}} > 0$. Then, we know that there is an $i \in \{1, \ldots, N\}$ with $Y_i \le M$. In particular, this claim cannot be part of S_{lc} and there is zero probability that we have v claims with $Y_i > M$. This explains why S_{sc} and S_{lc} cannot be independent. However, note that with the Poisson distribution as claims count distribution such a split in small and large claims leads to independent compound distributions, see Theorem 2.14 of the lecture notes (version of March 20, 2019).